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HISTORY AND BIOGRAPHY

See also 5156.

4877:

Frenkian, Aram. Recherches de mathématiques suméro-akkadiennes, égyptiennes et grecques. III. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 5–18.

[For parts I and II, see same Bull. **1 (49)** (1957), 17–32, 281–294; MR **20** #2244; **21** #2564.] In this final installment the author finishes his review of ancient mathematics with sections on Sumero-Akkadian numerical tables, algebra, and geometry. He concludes that the influence of the Orient, like that of the Greeks, was decisive and significant in the formation of modern science.

E. S. Kennedy (Beirut)

4878:

Czwalina, Arthur. Die Geometrie des Ptolemaeus von Alexandria. Enseignement Math. (2) **4** (1958), 292–299.

Following closely the original, the author shows, in modern notation, the trend of thought in Ptolemy's proof of his basic theorem:

$$\sin(CE)/\sin(EA) = \sin(CZ)/\sin(ZD)\cdot\sin(DB)/\sin(BA),$$

where ABC is a spherical triangle, given in his "Mathematical Syntaxis" (13 books). He then elucidates the analogy of this theorem (Ptolemy's main tool for astronomical computations) with Menelaus' theorem (on the transversal of a plane triangle). Moreover, the author shows in a special case how Ptolemy solves problems of spherical trigonometry, utilizing the "spherical theorem of Menelaus". Ptolemy reduces it all to rectangular triangles.

S. R. Struik (Cambridge, Mass.)

4879:

Davis, Natalie Zemon. Sixteenth-century French arithmetics on the business life. J. Hist. Ideas **21** (1960), 18–48.

4880:

Sierpiński, Wacław. The Warsaw school of mathematics and the present state of mathematics in Poland. Polish Rev. **4** (1959), no. 1–2, 1–13.

A lecture delivered on April 15, 1959, at the Polish Institute of Arts and Sciences in America. A non-technical factual account of activities since before World War I.

4881:

★Hardy, G. H. Ramanujan: twelve lectures on subjects suggested by his life and work. Chelsea Publishing Company, New York, 1959. iii+236 pp. (1 plate)

Originally published by Cambridge Univ. Press, England, and Macmillan, N.Y., 1940 [MR **3**, 71].

4882:

Smithies, F. John von Neumann. J. London Math. Soc. **34** (1959), 373–384.

Scientific and personal biography, with selected bibliography.

4883:

★Eulerus, Leonhardus. Opera omnia. Series secunda. Opera mechanica et astronomica. Vol. XIII. Commentationes mechanicae ad theoriam corporum fluidorum pertinentes. Vol. posterius. Edidit Clifford Ambrose Truesdell. Societatis Scientiarum Naturalium Helveticae, Lausanne, 1955. cxviii+375 pp. $12.50.

This volume contains Euler's Treatise on Fluid Mechanics (1766). The first three sections of this treatise are mainly a remodelling of earlier papers of Euler. The highly interesting section IV, "On the motion of air in tubes", deals with acoustics. The first two chapters of section IV contain a fuller treatment of the subject matter of an earlier paper, "More detailed enlightenment on the generation and propagation of sound ...", but the last four chapters are almost entirely new. Truesdell's excellent Introduction contains a highly instructive and reliable account of the historical development of the theory of aerial sound from 1687 to 1788. A very good German translation of Euler's Treatise on Fluid Mechanics was published by Brandes in 1805.

B. L. van der Waerden (Zürich)

4884:

★Haar, Alfred. Gesammelte Arbeiten. Herausgegeben von B. Szőkefalvi-Nagy. (Part in Hungarian) Verlag der Akademie der Wissenschaften, Budapest, 1959. 661 pp. (1 plate) 250 Ft.; $14.00.

Alfred Haar war ein würdiger Vertreter der grossen ungarischen Tradition in Mathematik. Trotz der Kürze seines Lebens (1885–1933) ist es ihm gelungen, bahnbrechende Entdeckungen in drei verschiedenen Gebieten der Mathematik zu machen und eine Reihe anderer wichtiger Arbeiten zu schreiben. Die Herausgabe seiner gesammelten Arbeiten von der Ungarischen Akademie der Wissenschaften, 26 Jahre nach seinem Tode, ist deshalb sehr begrüssenswert.

Die 35 veröffentlichten Arbeiten von Alfred Haar erscheinen hier, nach Gegenständen geordnet: (A) Eine Jugendarbeit mit D. König über einfach geordnete Mengen. (B) Orthogonalfunktionenreihen. Hier findet man das berühmte Haarsche Funktionensystem, das in Haars Inauguraldissertation eingeführt wurde, und eine Reihe von Mitteilungen über verschiedene Entwicklungsprozesse. (C) Analytische Funktionen. (D) Partielle Differentialgleichungen. (E) Variationsrechnung. Hier findet man in 7 Arbeiten das "Haarsche Lemma" und seine Anwendung auf verschiedene Probleme der Variationsrechnung. (F) Approximation von Funktionen und lineare Ungleichungen. Haar's einzige Arbeit über gleichmässige Approximation war der Ursprung einer grossen Reihe von Arbeiten über Approximation, insbesondere von der russischen Schule. (G) Diskrete Gruppen und Funktionenalgebren. Die Fragen über unendliche Gruppen und isomorphe Funktionenfolgen, die Haar betrachtete, werden heutzutage meistens mit der Pontrjaginschen Dualitätstheorie behandelt. Seine Arbeit über Gruppencharaktere ist auch heute noch von sachlichem Interesse. (H) Kontinuierliche Gruppen. In einer einzigen Arbeit hat Haar die Bahn für die ganze Theorie der lokalkompakten Gruppen und abstrakten harmonischen Analysis geöffnet, und zwar durch die Entdeckung des invarianten Masses auf diesen Gruppen. Heutzutage ist der Begriff des Haarschen Masses eines der wichtigsten Hilfsmittel der Analysis, und die Phrase "Haarsches Mass" ist ein Ausdruck, der von allen Analytikern verstanden wird.

Die meisten von Haars Arbeiten sind im Original in deutscher Sprache erschienen. Diese Arbeiten erscheinen in den Gesammelten Arbeiten ohne Änderung. Den in Ungarisch erschienen Originalarbeiten ist, mit Ausnahme eines Berichtes über Bolyais Geometrie, in den Gesammelten Arbeiten eine deutsche Übersetzung beigefügt. Der Herausgeber, Béla Szökefalvi-Nagy, hat wertvolle Kommentare zu jeder Gruppe von Arbeiten geschrieben und auch die Übersetzungen der ungarischen Arbeiten ins Deutsche verfasst. *E. Hewitt* (Seattle, Wash.)

GENERAL

See also 5415.

4885:

★Rédei, Ladislaus. Algebra. Erster Teil. Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Bd. 26, Teil 1. Akademische Verlagsgesellschaft, Geest & Portig, K.-G., Leipzig, 1959. xv + 797 pp. DM 48.00.

Ce traité fait partie d'une collection dont le but est la publication d'ouvrages de mathématiques appliquées à la physique et à la technique. C'est à dire que l'auteur s'est assigné pour tâche de s'en tenir aux seules structures ayant une importance dans la pratique. Depuis la publication de l'édition hongroise [Akadémiai Kiadó, Budapest, 1954; cf. l'analyse détaillée par chapitres, MR **16**, 559], beaucoup d'additions et de perfectionnements ont été apportés à ce premier volume. Mais comme il n'était pas possible d'adopter comme matière d'enseignement tous les résultats obtenus en algèbre depuis les dix dernières années, l'auteur a dû renoncer à être complètement exhaustif. Par exemple, dans la théorie des groupes, au lieu d'énumérer une longue série de propositions spéciales, il a dégagé un nombre raisonnable de théorèmes fondamentaux et très généraux. Plusieurs paragraphes, surtout dans les chapitres I et II, ont été disposés autrement, ou même dédoublés. Les fautes d'impression ont été corrigées et de nouveaux paragraphes ajoutés ou complétés, tant pour aligner l'ouvrage sur les résultats importants les plus récents de la littérature que pour faire profiter le lecteur des derniers travaux de l'auteur. Tels sont, entre autres, No. 23 [Abh. Math. Sem. Univ. Hamburg **22** (1958), 201–214; MR **20** #3856], No. 52 et 54 [Acta. Math. Acad. Sci. Hungar. **5** (1954), 169–195; MR **17**, 342], No. 92 [ibid. **6** (1955), 27–40; MR **17**, 343], No. 95 [ibid., 5–25; MR **17**, 344], No. 96 [Acta Math. Sci. Szeged **19** (1958), 98–126; Acta Math. Acad. Sci. Hungar. **6** (1955), 271–279; MR **20** #3845; **18**, 15], No. 120 [Acta Math. Sci. Szeged **17** (1956), 198–202; MR **18**, 462]. Un XIIième chapitre, remplissant 54 pages, a été ajouté. Il contient trois paragraphes, consacrés aux structures finies non commutatives, dont les sous-structures vraies de même espèce sont toutes commutatives (endliche einstufige nichtkommutative Strukturen). Par exemple tous les sous-groupes propres du groupe des quaternions sont abéliens. Le problème est complètement traité dans les trois cas où on sait le résoudre: No. 183, groupes, No. 184, anneaux, No. 185, groupoïdes associatifs. L'auteur fait une synthèse des travaux de ses davanciers et de ses propres recherches [Acta Math. Acad. Sci. Hungar. **8** (1957) 401–442; Acta Math. Sci. Szeged **19** (1958), 127–128; MR **20** #59, #3915; partiellement, Publ. Math. Debrecen **4** (1956), 303–324; MR **18**, 12]; il complète ces résultats et en donne un exposé d'ensemble. Un théorème donne, dans chacun des trois cas, la définition générale des solutions, le mode de génération, l'ordre de celle-ci et le détail de ses éléments. Chacun des trois paragraphes est suivi d'exemples et de problèmes à résoudre. Presque tous les paragraphes anciens ont également été assortis d'exemples nouveaux et de questions proposées; un grand nombre d'applications de l'algèbre à la théorie des nombres sont exposées. Quelques innovations concernent les notations et la terminologie (Faktorring, Hauptpolynom, ...). Le groupe symétrique est appelé "volle Permutations-Gruppe" suivant une locution déjà adoptée par G. Scorza [*Gruppi astratti*, Edizioni Cremonese, Rome, 1942; MR **10**, 588; p. 13]. En haut de chaque page on trouve, avec le titre du paragraphe, le nombre de celui-ci et, en bas, une flèche chiffrée indiquant combien de pages il faut tourner pour trouver le théorème le plus voisin.

On peut regretter que les structures non associatives aient été reléguées au rang d'accessoires et il faut espérer, dans la mesure où le parti-pris de s'en tenir aux matières conduisant à des applications pratiques le permettra, qu'une étude systématique de tels systèmes sera développée dans les volumes suivants. Au cours de l'ouvrage les sources ne sont pas citées car, comme le livre est avant tout destiné à l'enseignement, il n'était pas utile de donner dans le texte des références détaillées. D'ailleurs, il est toujours facile de reconstituer ces références assez rapidement en recourant aux mémoires originaux mentionnés dans la bibliographie. Si ce traité est, d'abord, un ouvrage didactique, c'est aussi un excellent instrument de documentation et de travail pour les mathématiciens qui ne s'occupent pas spécialement d'algèbre. On y sent à chaque ligne la préoccupation constante de venir en aide au lecteur. Que celui-ci soit un étudiant ou une personne appelée à lire accidentellement un mémoire d'algèbre, il

trouvera très rapidement dans le livre, grâce à un index de plus de mille mots, renvoyant à des exposés autonomes, les éclaircissements dont il aura besoin. L'auteur s'est donné la peine de numéroter les paragraphes dans l'ordre naturel et de dresser le lexique avec renvoi direct aux pages, ce qui, étant donnée l'ampleur du volume, représente un travail considérable, consenti pour éviter des recherches pénibles. Comme l'auteur domine de haut son sujet et comme son but est de servir, non d'étonner, il a pu offrir au lecteur une œuvre fortement charpentée mais claire, agréable à lire, et où les définitions sont parfaitement rigoureuses, mais commodes, directes et sans pédanterie. Il serait vivement souhaitable que cet ouvrage imposant, profond et d'une si grande efficacité fut traduit en français et publié dans un proche avenir.

A. Sade (Marseille)

4886:

★**Structures algébriques et structures topologiques.** Monographies de l'Enseignement Mathématique, No. 7. Institut de Mathématiques, Université de Genève; Association des Professeurs de Mathématiques de l'Enseignement Public, Paris; 1958. 198 pp. 20 francs suisses.

A collection of lectures held in 1956–57 at the Institut Henri-Poincaré, sponsored jointly by the Société Mathématique de France and the Association des Professeurs de Mathématiques de l'Enseignement Public, for the purpose of strengthening the contact between lower and higher echelons of education and infusing the modern spirit into the teaching of mathematics. The lectures were published individually in the Bulletin of the Association. The topics and lecturers are: Algebraic structures (H. Cartan); Rings, congruences, ideals (P. Dubreil); Vector spaces, linear forms and equations (G. Choquet); Linear maps and matrices (A. Lichnerowicz); Quadratic and hermitian forms (P. Lelong); Classical groups (L. Lesieur); Projective spaces (A. Revuz); The real line (G. Choquet); Euclidean and metric spaces (A. Revuz); Concepts related to metric space structure (G. Choquet); Function spaces and modes of convergence (J. Dixmier); Concepts of general topology (Ch. Pisot); Compact and locally compact spaces (Ch. Pisot); Topological groups and vector spaces (R. Godement); The concept of dimension (H. Cartan); Coverings and fundamental group (J.-P. Serre); Elements of homology (L. Schwartz).

The lectures are more or less independent but the collection is coherent, and the presentation, while introductory and sometimes sketchy, is technical, brisk, and explicit. Some misprints; e.g., on p. 67 in definition of hermitian scalar product: "$a \cdot b = b \cdot a$ (non-commutativité)".

4887:

★**Levi, Howard. Elements of algebra.** 3rd ed. Chelsea Publishing Co., New York, 1960. xi+161 pp.

Unchanged (apparently) from first edition [MR **16**, 325].

4888:

★**Kemeny, John G.; Mirkil, Hazleton; Snell, J. Laurie; and Thompson, Gerald L. Finite mathematical structures.** Prentice-Hall, Inc., Englewood Cliffs, N.J., 1959. xi+487 pp. $7.95.

Dieses ausgezeichnete Buch ist geeignet, Begeisterung für die Mathematik bei einem Leser hervorzurufen, dessen Hauptfach nicht die Mathematik ist, der aber etwas Mathematik in seinem Beruf braucht. Es will von dem Leser in ernster und aufmerksamer Weise benützt werden, macht ihm aber das Studium angenehm und verhältnismässig leicht, indem es klar, deutlich, durchsichtig und interessant geschrieben ist und viele Beispiele aus dem täglichen Leben bringt. Auch sind am Schluss eines jeden Paragraphen viele Übungen gebracht.

Der zentrale Begriff des Buches ist der der Aussage. Damit eine Aussage sinnvoll ist, muss bereits ein Raum logischer Möglichkeiten vorhanden sein, relativ zu dem die Behauptung der Aussage auf ihren Wahrheitsgehalt zu prüfen ist. Eine Aussage ist also nicht für sich allein w (wahr) oder f (falsch), sondern nur relativ zu einem gewissen Raum logischer Möglichkeiten, je nachdem auf welche Möglichkeit sie bezogen wird, manchmal w und manchmal f. Damit entfällt die Kritik, die Gale (in seinem Referat [MR **18**, 860] über den Vorgänger des vorliegenden Buches [*Introduction to finite mathematics*, Prentice-Hall, Englewood Cliffs, N.J., 1957]) an dem Begriff der Aussage der Verfasser geübt hat. Es besteht nämlich nach dem Gesagten kein Widerspruch zwischen dem Satz "Die fundamentale Eigenschaft einer jeden Aussage ist, dass sie entweder w oder f ist" und dem Satz: "Normalerweise wird es, für eine gegebene Aussage, viele Fälle geben, in denen sie w ist und viele Fälle, in denen sie f ist." Der erste Satz gilt nämlich relativ zu einem vorgegebenen Raum logischer Möglichkeiten.

Insbesondere steht der Begriff der Aussage auch im Zentrum des Kapitels über die Wahrscheinlichkeitsrechnung. Die Verfasser sprechen von der Wahrscheinlichkeit einer Aussage, worunter sie das (Wahrscheinlichkeits-) Mass derjenigen Teilmenge des Raumes der logischen Möglichkeiten verstehen, deren Elemente diese Aussage bewahrheiten (der "Wahrheitsmenge" der Aussage). Die Einführung des Begriffs der Aussage in die Wahrscheinlichkeitsrechnung und in die Statistik ist von den Verfassern in überaus glücklicher Weise beschrieben. Überhaupt ist das Kapitel über die Wahrscheinlichkeitsrechnung eine sehr schöne Darstellung der Grundzüge dieser Disziplin.

Im übrigen gelten für das Buch die positiven Urteile von Gale über das vorhergehende Buch. Der Leser möge also das oben zitierte Referat nachschlagen. Der Unterschied zwischen diesen beiden Büchern besteht darin, dass während das vorhergehende Buch für Biologen und Sozialwissenschaftler bestimmt ist, das gegenwärtige Buch mehr den mathematischen Bedürfnissen des Physikers und Ingenieurs entgegenkommt. Dementsprechend setzt das vorliegende Buch eine grössere mathematische Reife des Lesers voraus und bringt weitergehende Ergebnisse.

Die Benennung "Endliche mathematische Strukturen" im vorliegenden Buch ist nicht darauf zurückzuführen, dass die Verfasser den Begriff der Grenze vermeiden, sondern darauf, dass die in diesem Buch behandelten "unendlichen" Probleme endliche Gegenpartner besitzen, die in voller Ausführlichkeit behandelt werden, bevor der Sprung ins Unendliche gemacht wird (aus dem Vorwort).

Der Inhalt des Buches ist folgender: Das erste Kapitel handelt von der elementaren symbolischen Logik und bringt eine Anwendung auf die elektrischen Netze. Das zweite Kapitel ist der Mengenlehre gewidmet. Es dringt bis zur Theorie der Funktionen auf einer Menge und insbesondere der numerischen Funktionen vor. Diese

beiden Kapitel bilden die Grundlage für alles folgende. Kapitel 3 handelt von der Wahrscheinlichkeitsrechnung. Kapitel 4 behandelt die elementare lineare Algebra. Kapitel 5 handelt von den konvexen Mengen im Zusammenhang mit den linearen Programmierungsproblemen. Kapitel 6 ist den endlichen Markovschen Ketten gewidmet. Kapitel 7 bringt die kontinuierliche Wahrscheinlichkeitstheorie, mit einer Anwendung auf die Fehlertheorie.

B. Germansky (Berlin)

4889:

★**Goursat, Édouard. A course in mathematical analysis: Vol. 1: Derivatives and differentials, definite integrals, expansion in series, applications to geometry. Vol. 2, Part 1: Functions of a complex variable. Vol. 2, Part 2: Differential equations.** Translated by E. R. Hedrick (Vol. 1), and E. R. Hedrick and O. Dunkel (Vol. 2). Dover Publications, Inc., New York, 1959. ix + 548 + x + 259 + viii + 300 pp. Paperbound. Vol. 1, $2.25; Vol. 2, $1.65 per part; $5.00 the set.

Unaltered republication of the translation published by Ginn and Co., Boston-New York; Vol. I (1904) was translated from the first French edition [Gauthier-Villars, Paris, 1902], and Vol. II (1916) from the second French edition [Gauthier-Villars, Paris, 1915].

4890:

★**Sokolnikoff, I. S.; and Redheffer, R. M. Mathematics of physics and modern engineering.** McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. ix + 810 pp. $9.50.

This book is a successor to, not a new edition of, the well-known textbook by I. S. and E. S. Sokolnikoff, *Higher mathematics for engineers and physicists* [McGraw-Hill, New York], the first edition of which was published in 1934 and the second in 1941. It contains nearly 50% more material than its predecessor and reflects the increased sophistication of present-day applicable mathematics. The chapter headings are: (1) Ordinary differential equations; (2) Infinite series; (3) Functions of several variables; (4) Algebra and geometry of vectors. Matrices; (5) Vector field theory; (6) Partial differential equations; (7) Complex variable; (8) Probability; (9) Numerical analysis. The appendix contains: (A) Determinants; (B) The Laplace transform; (C) Comparison of the Riemann and Lebesgue integrals; (D) Table of the error integral.

4891:

★**Condon, E. U.; and Odishaw, Hugh. (Editors) Handbook of physics.** McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. xxvi + 1459 pp. $25.00.

This is a one volume handbook in which the editors have sought to include "what every physicist should know". Of the 1500 pages (3 years at 10 pages a week), 165 are devoted to mathematics. These are divided into 13 chapters of which the titles are: arithmetic, algebra, analysis, ordinary differential equations, partial differential equations, integral equations, operators, geometry, vector analysis, tensor calculus, calculus of variations, elements of probability, statistical design of experiments. There is a list of references at the end of each article. The book is clearly printed and sturdily bound.

A listing of the rest of the subjects covered by the volume would take too much space. The exposition is elementary and clear. The apportionment of space seems reasonable except that $1\frac{1}{2}$ pages for Hilbert space and linear operators seems rather skimpy when the calculus of variations gets 10. To the reviewer it seems that the primary usefulness of this volume is for information on subjects about which the reader is rather ignorant. (The specialist in algebraic number theory or quantum field theory will find the articles on glass or surface tension and absorption just his speed.) In brief it is a sort of systematic account of the basic facts of physics at the level of the Scientific American but with equations. In the reviewer's opinion the editors have achieved their objective.

A. S. Wightman (Princeton, N.J.)

4892:

Anonymous. Bibliography of Polish Mathematics for the ten years 1944–1954. Wiadom. Mat. (2) **2** (1957), 1–154. (Polish)

This bibliography of some 1700 entries lists, according to subject matter, the mathematical publications, including monographs, textbooks and university mimeographed notes, by Polish citizens, published or accepted for publication in and outside Poland during the period from 22 July 1944 to 31 December 1954.

4893:

Pogorzelski, H. A. A note on mathematical notation. Math. Mag. **33** (1959/60), 24.

The author suggests the use of the symbols $\upharpoonleft$, $\downharpoonright$ and $\updownarrow$ for writing subscripts and superscripts on the main line of the text, e.g., $B\upharpoonleft n\upharpoonleft 2$ or $B^{n\upharpoonleft 2}$ could replace B^{n^2}.

P. Samuel (Clermont-Ferrand)

4894:

Montel, Paul. La mathématique méditerranéenne. Bull. Soc. Math. France **86** (1958), 257–270.

A congress of mathematicians of Latin 'expression' was held at Nice, France, in 1957, on the suggestion of Sansone. The author of the present article espouses the idea of meetings by mathematicians speaking a language of Latin origin as opposed to the diverse languages spoken at usual congresses because the language difficulty is removed. He also believes that the spirit of mathematics differs from one language group to another. Thus he believes the French language best for the expression of abstract concepts (though here he opposes it to Latin); the 'Latin group' enjoys the aesthetic side of mathematics more, and is probably more inclined to think geometrically. His major reason for advocating congresses of 'Latin' mathematicians is that they surround the Mediterranean, which was the geographical scene of the origin and early development of mathematics. To amplify this statement he reviews briefly the early history of mathematics among the Egyptians, Babylonians, Greeks and Arabs.

M. Kline (New York, N.Y.)

LOGIC AND FOUNDATIONS

4895:

★**Skolem, Th. Une relativisation des notions mathématiques fondamentales.** Le raisonnement en mathémati-

ques et en sciences expérimentales, pp. 13–18. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

To the current conception of mathematics which he describes as absolutist and platonist, the author opposes a relativist conception of its fundamental notions, based upon the idea of a formal system. Formal systems are used to make the notions and methods of mathematics precise. According to this relativist conception, Cantor's transfinite cardinal numbers cannot have an absolute sense. In the author's opinion (for which he also refers to H. Poincaré), there exist no sets non-denumerable in an absolute sense; in other words, the introduction of such sets is useless.—The talk was followed by a spirited discussion, in which Tarski, Krasner, Mostowski, and de Possel took part. *E. W. Beth* (Amsterdam)

4896:

★**Mostowski, Andrzej. Quelques observations sur l'usage des méthodes non finitistes dans la métamathématique.** Le raisonnement en mathématiques et en sciences expérimentales, pp. 19–32. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

The results discussed here require meta-mathematical ideas both in their statements and in their proofs. The number of constants and the number of axioms need not be denumerable, so that non-finitist methods come into play.

The first section concerns set-theory and the arithmetic of real numbers. It sketches the construction of a statement of the form "$(x)(\exists y)((x, y) \in B)$", where B is a certain Borel subset of the plane and x and y are real numbers, and such that not only is this statement undecidable in the arithmetic of reals, but rather powerful machinery must be added before it becomes decidable.

Another group of results concerns automorphisms of models. It turns out that first-order systems have models whose group of automorphisms is very large. Closely related to this are the results on systems which are categorical in a given power. There are many problems here which are still open, even when the axioms are the Peano-axioms. The author summarizes the present state of the problems and gives references.

Finally there is a discussion of a more general notion of quantifiers including for instance "there are at most $\aleph_0$ elements such that ...". If these are also allowed many meta-mathematical results, such as the Skolem-Löwenheim theorem, fall by the wayside. This again leaves many open questions. *L. N. Gál* (New Haven, Conn.)

4897:

★**Martin, Roger. Sur les notions intuitives mises en œuvre par la constitution et l'étude d'un système formel.** Le raisonnement en mathématiques et en sciences expérimentales, pp. 91–96. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

The development of the study of formal systems does not deprive of their former interest those intuitive elements which logical analysis brings to light, but provides them rather with a particular interest in two widely different cases: (i) that of those "weak" intuitions which lend a maximal solidity to our constructions (as, for instance, in the theory of recursive functions), and (ii) that of those "strong" (and improperly so called) intuitions which compel us to abandon those zones of thought traditionally considered safe but which permit us to achieve new results. In either case, such intuitions can be used only after they have been given a mathematical expression. Both for recursive functions and for non-finitist metamathematics, the underlying intuitions receive a precise sense only if we succeed in translating them into a symbolic discourse which already belongs to mathematics. *E. W. Beth* (Amsterdam)

4898:

★**Gonseth, F. Sur la méthodologie des recherches sur les fondements des mathématiques.** Le raisonnement en mathématiques et en sciences expérimentales, pp. 97–107. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

On the basis of a broad survey of the development of foundational research, the author defends the conception of a "méthodologie d'élémentarité et de cheminement au sens de la théorie ouverte de la connaissance". *E. W. Beth* (Amsterdam)

4899:

★**Feys, Robert. Expression de la vérifiabilité expérimentale dans le raisonnement formalisé.** Le raisonnement en mathématiques et en sciences expérimentales, pp. 109–115. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

Gödel's well-known interpretation of intuitionistic sentential logic within S4 is applied in the following manner. In S4, the classical sentential connectives are taken in their customary meaning whereas necessity is taken to express experimental verifiability; the meaning of possibility is then uniquely defined. In intuitionistic logic, the sentential connectives are taken likewise in their classical meaning, with the understanding, however, that their application is restricted to the assertion of verified facts. For instance, an intuitionistic implication $P \rightarrow Q$ is interpreted by the formula $CLPLQ$ of S4. *E. W. Beth* (Amsterdam)

4900:

★**Porte, J. Recherches sur les logiques modales.** Le raisonnement en mathématiques et en sciences expérimentales, pp. 117–126. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

The author considers modal logics from an algebraic viewpoint. He first constructs a modal logic which he considers to be the weakest possible system. He then develops stronger systems and compares his systems with the formal systems of Lewis. *A. Rose* (Nottingham)

4901:

★**Destouches, Jean-Louis; et Février, Paulette. Remarques sur certains aspects formels des théories physiques.** Le raisonnement en mathématiques et en sciences expérimentales, pp. 127–133. Colloques Internationaux du Centre National de la Recherche Scientifique, LXX. Editions du Centre National de la Recherche Scientifique, Paris, 1958. 140 pp. 1400 francs.

After an introductory discussion (formal structure of physical theories, equivalence of physical theories, complete and incomplete theories, general theory of predictions), the authors study the possibility of introducing quantifiers in quantum logic; two different procedures are described, the second of which leads back to the quantifiers of the classical calculus.

E. W. Beth (Amsterdam)

4902:

★**Guillaume, Marcel. Les tableaux sémantiques du calcul des prédicats restreint.** Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 153, 13 pp. Secrétariat mathématique, Paris, 1958. 189 pp. (mimeographed)

The method of semantic tableaux, as described by the reviewer, is developed in a more abstract manner, essentially using set-theoretic devices; the restriction to countable sets of formulas and symbols is no longer observed and, in addition, a tendency towards an algebraization is clearly expressed. By way of conclusion, the author states various remarks of a more general nature.

E. W. Beth (Amsterdam)

4903:

Prior, A. N. Epimenides the Cretan. J. Symb. Logic **23** (1958), 261–266.

Starting from L. Jonathan Cohen's remark [same J. **22** (1957), 225–232; MR **20** #4482] that the Epimenidean as contrasted with the Eubulidean version of the Liar paradox is the one that threatens logicians who attempt to formalise the use of indirect rather than direct discourse, the author sets out to fill in the lacuna in his own work [*Time and modality*, Oxford Univ. Press, New York, 1957] to which Cohen has drawn attention. He suggests that if Epimenides the Cretan says 'Nothing asserted by a Cretan is the case', and no Cretan says anything else, then it is indeed the case that nothing asserted by a Cretan is the case, but Epimenides has not asserted this.

To say such a thing is to suggest that the relations between direct speech and indirect are extremely tricky. On the other hand, if we are prepared to say such things, the logic of indirect speech in itself can be kept extremely simple. To formalise it, the system in the author's aforementioned book can stand, without any tinkering with its formation rules or transformation rules.

E. W. Beth (Amsterdam)

4904:

Lambek, Joachim. The mathematics of sentence structure. Amer. Math. Monthly **65** (1958), 154–170.

This paper is a substantial contribution to mathematical linguistics; in particular it follows the line of thought which tries to apply to natural languages the theory of grammatical categories constructed for formalized languages by Leśniewski [Fund. Math. **14** (1929), 1–81]. Ajdukiewicz [Studia Philos. **1** (1935), 1–27] and Bar-Hillel [Language **29** (1953), 47–58] are immediate precursors of Lambek's work. The goal is to find an algorithm in terms of grammatical types for distinguishing sentences from nonsentences, e.g., in English. Two primitive types are assumed, that of sentences (s) and that of nouns (n). If x and y are types, then so are x/y and $y\backslash x$; we assign x/y to an expression which forms an expression of type x if followed by an expression of type y; and we assign $y\backslash x$ to an expression which forms an expression of type x if preceded by an expression of type y. Then we assume the following rules of computing types:

(i) $(x/y)y\to x,\quad y(y\backslash x)\to x;$

(ii) $(x\backslash y)/z \leftrightarrows x\backslash(y/z);$

(iii) $(x/y)(y/z)\to x/z,\quad (x\backslash y)(y\backslash z)\to x\backslash z;$

(iv) $x\to y/(x\backslash y),\quad x\to(y/x)\backslash x.$

A syntactic calculus is formed, based on $x\to x$, $(xy)z\to x(yz)$, and $x(yz)\to(xy)z$ as axioms and on the following rules of inference: if $xy\to z$, then $x\to z/y$ and $y\to x\backslash z$; if $x\to z/y$, then $xy\to z$; if $y\to x\backslash z$, then $xy\to z$; if $x\to y$ and $y\to z$, then $x\to z$. The rules (i)—(iv) are then deduced and it is shown that there is a decision procedure for the problem of whether $x\to y$ is deducible in the syntactic calculus (similar to that of Gentzen [Math. Z. **39** (1934), 176–210, 405–431] for intuitionistic sentential calculus). An analogy with residual lattices is noticed.

This paper shows some progress over previous work, e.g., it considers not only the right-hand arguments but also left-hand arguments of an expression figuring as a functor. Following Curry's theory of functionality all functors here are unary.

H. Hiż (Philadelphia, Pa.)

4905:

Stahl, Gerold. An opposite and an expanded system. Z. Math. Logik Grundlagen Math. **4** (1958), 244–247.

Various systems of formulas of the propositional calculus are considered, along with some of their simple properties, such as certain types of consistency and completeness. One such system is the set of contravalid formulas, which is stated to be identical with a class of "duals" of tautologies.

E. Mendelson (New York, N.Y.)

4906:

Hasenjaeger, Gisbert. Zur Axiomatisierung der k-zahlig allgemeingültigen Ausdrücke des Stufenkalküls. Z. Math. Logik Grundlagen Math. **4** (1958), 175–177.

It has been shown by Asser and Schröter [Math. Nachr. **19** (1958), 73–86; MR **21** #2581] that in monadic type-theory the set of k-valid formulas can be axiomatized by a certain class of universally valid statements together with the precisely k-valid formula $\exists_k{}^0!!$, which asserts that there are exactly k objects of lowest type. (A formula is said to be precisely k-valid if it holds just in those domains with k objects of lowest type.) The author shows that $\exists_k{}^0!!$ cannot be replaced as an axiom by an arbitrary precisely k-valid formula. In fact, he proves that, for any precisely k-valid formula H, not-$\vdash H\to\exists_k{}^0!!$ if and only if there is a universally valid, unprovable formula θ such that $\vdash H\leftrightarrow(\theta\to\exists_k{}^0!!)$. The result then follows from Gödel's

undecidability theorem. The method of proof is a simple and elegant application of the theorem of Asser and Schröter. *E. Mendelson* (New York, N.Y.)

4907:

Hasenjaeger, G. Über Interpretationen der Prädikatenkalküle höherer Stufe. Arch. Math. Logik Grundlagenforsch. **4** (1958), 71–80.

The author deals with the so-called nonmaximal (nichtmaximalen) interpretations of (unramified) higher predicate calculi in the following sense. Let G be a group of permutations of the class of individuals $D(x)$, and G_f a subgroup of G which leaves invariant the finite sequence f of individuals. The author shows that all the sets X^s (subsets of the Cartesian product of s factors $D(x)$) invariant under some G_f form a range of the s-placed predicate variables of the calculus (PK_2) (of second order). (The X^s in question form a family closed under the obvious operations as imposed by the recursive formation rules.) An extension to Church's simple theory of types is outlined. Some interesting examples illustrate the method, the idea of which, in fact, goes back to Fraenkel [see S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. **1922**, 253–257].

L. Rieger (Prague)

4908:

Grzegorczyk, A.; Mostowski, A.; and Ryll-Nardzewski, C. The classical and the ω-complete arithmetic. J. Symb. Logic **23** (1958), 188–206.

The authors consider two formalisations of the theory of natural numbers. Both systems are applied second order functional calculi. The second formalisation is obtained from the first by adjoining a non-finitary primitive rule of procedure. In each case a number of properties of the system are established, including theorems concerning various types of undecidability.

A. Rose (Nottingham)

4909:

Craig, W.; and Vaught, R. L. Finite axiomatizability using additional predicates. J. Symb. Logic **23** (1958), 289–308.

Eine (in der elementaren Logik mit Gleichheit formalisierte) Theorie T heißt "mit zusätzlichen Prädikaten endlich axiomatisierbar" (e.a.$^+$), wenn es eine endlich axiomatisierbare (e.a.) Theorie T' gibt, so daß (1) die Sprache von T' die Sprache von T umfaßt, (2) jede Aussage von T genau dann in T' gilt, wenn sie in T gilt. T heißt dagegen "semantisch e.a.$^+$" (s.e.a.$^+$), wenn es eine e.a. Theorie T' gibt mit (1) und (2^s) jede Interpretation ("possible realization") von T läßt sich genau dann zu einem Modell von T' fortsetzen, wenn sie ein Modell von T ist. Nach dem Gödelschen Vollständigkeitssatz gilt: s.e.a.$^+$ impliziert e.a.$^+$, ferner ist bekannt, daß e.a.$^+$ die (rekursive) Axiomatisierbarkeit impliziert.—Verff. beweisen, wie von Craig früher angekündigt, daß auch s.e.a.$^+$ von e.a.$^+$ impliziert wird. Haupthilfsmittel ist eine Verschärfung des Kleeneschen Satzes, daß eine axiomatisierbare Theorie, die nur unendliche Modelle hat, e.a.$^+$ ist. Kleenes Beweis liefert nämlich auch die Gültigkeit der Bedingung (2^s), wenn diese auf abzählbare Interpretationen eingeschränkt wird. Durch eine Modifikation des Kleeneschen mengentheoretischen Beweises ergibt sich die uneingeschränkte Bedingung (2^s). Ist nun T e.a.$^+$, so ist T^∞ (definiert dadurch, daß die Klasse der in T gültigen Aussagen erweitert wird durch Hinzunahme der Aussagen $\neg\,\delta_n$ ($n=0, 1, 2, \cdots$), wobei δ_n besagt, daß es genau n verschiedene Elemente gibt) axiomatisierbar (weil T axiomatisierbar ist) und daher—nach obigem—s.e.a.$^+$. Hieraus folgt leicht, daß auch T s.e.a.$^+$ ist.—Die von Ehrenfeucht angegebenen Theorien (keine Prädikate, und als Axiome $\neg\,\delta_n$ für die n aus einer rekursiv aufzählbaren, aber nicht rekursiv entscheidbaren Klasse) sind axiomatisierbar, aber nicht e.a.$^+$.—Für die Theorien, die in der elementaren Logik ohne Gleichheit formalisiert sind, folgt aus diesen Ergebnissen, daß eine axiomatisierbare Theorie stets e.a.$^+$ ist, aber nicht stets s.e.a.$^+$. *P. Lorenzen* (Kiel)

4910:

Gilmore, P. C. An addition to "Logic of many-sorted theories". Compositio Math. **13**, 277–281 (1958).

Herbrand [*Recherches sur la théorie de la démonstration*, Warsaw, 1930; Ch. 3, 3.4—3.42] first stated, with an erroneous proof, the equivalence of suitable one-sorted theories to given many-sorted theories. A. Schmidt supplied a detailed proof in (1) Math. Ann. **115** (1938), 485–506 which, however, has a gap, as pointed out by Bernays. This gap was eliminated by Schmidt and a supplement was eventually published in (2) Math. Ann. **123** (1951), 187–200 [MR **13**, 614]. A simpler proof of the same result was included in a paper of the reviewer's [J. Symbolic Logic **17** (1952), 105–116; MR **14**, 3], which credits the first correct proof to (1) but fails to mention (2) because the paper had been completed before (2) appeared. Both proofs are subject to the restriction that in each argument place of the primitive predicates of the many-sorted theories may occur only variables of a given sort. The author now extends the reviewer's proof to give a stronger result which removes the above restriction. The extension calls for rather elaborate argumentation.

Hao Wang (Murray Hill, N.J.)

4911:

Krasner, Marc. Théorie de la définition. II. Théorie des catégories supérieures. Systèmes non kroneckériens. Origine et solution des paradoxes. La signification du définitionnisme. J. Math. Pures. Appl. (9) **37** (1958), 55–101.

Verf. führt mit dem vorliegenden Teil seine Skizze [dasselbe J. **36** (1957), 325–357; MR **19**, 935] eines neuen Lösungsversuchs der mengentheoretischen Paradoxien zu Ende. Ausgangspunkt des "Definitionismus" des Verf. ist die Forderung Lebesgues, nur "objets nommables" in der Mathematik zuzulassen. Verf. gibt eine formalisierte Theorie an, die dieser Forderung genügt. Er betrachtet die Anerkennung der beschränkt-endlichen, der unbeschränkt-endlichen, der abzählbaren und der überabzählbaren Mengen (z.B. der Mächtigkeiten $2^{\aleph_0}$ und $2^{2^{\aleph_0}}$) als gleichberechtigte Standpunkte. Er verlangt daher nicht, daß die Mittel zur Aufstellung der formalisierten Theorien finit sind: es werden wahlweise auch "Ausdrücke" mit abzählbar oder überabzählbar vielen Einzelzeichen zugelassen. Derjenige Teil des Definitionismus, der sich auf endliche Ausdrücke beschränkt, hat aber natürlich—vom konstruktiven Standpunkt—allein Interesse. Er wird auch vom Verf. besonders eingehend als "système kroneckérien" (SK) behandelt.

In SK werden nur endliche Mengen als "aktuale"

Eigenschaften zugelassen; alle unendlichen Mengen sind "virtuelle" Eigenschaften. Eine Unterscheidung zwischen "Menge" und "Eigenschaft" wird vom Verf. als scholastisch verworfen. Der Aufbau des Systems beginnt mit der Konstruktion von "Definitionsprädikaten" $P(x)$, die genau ein Objekt x charakterisieren sollen. Dazu geht Verf. von den folgenden primitiven Formeln aus: $x=y$, $x \neq y$, $x \in y$, $x \notin y$ mit der üblichen Bedeutung; $y \swarrow x$ [resp. $y \searrow x$] (x ist ein geordnetes Paar mit dem ersten [resp. zweiten] Glied y); $I(x)$ (x ist ein Individuum); $a(x)$ (x ist eine aktuale Eigenschaft); $v(x)$ (x ist eine virtuelle Eigenschaft); $a(Cx)$ [resp. $v(Cx)$] (das Komplement von x ist eine aktuale [resp. virtuelle] Eigenschaft). Außerdem stehen beliebige Namen für Individuen zur Verfügung. Alle Definitionsprädikate werden dann mit Hilfe von & und den beiden Quantoren (x), $(\exists x)$ nach bestimmten Regeln zusammengesetzt. Um diese Regeln übersichtlich angeben zu können, benützt Verf. eine zweidimensionale Notation, auf die hier nicht eingegangen werden kann. Jedes Definitionsprädikat wird nach ihr durch einen "Baum" dargestellt, aus dem zu entnehmen ist, ob das zu definierende x ein Individuum (evtl. welches), ein Paar oder eine Menge ist—und in den letzten beiden Fällen enthält der Baum auch Definitionen der Glieder bzw. Elemente.

Der Witz der definitionistischen Systeme besteht nun darin, daß für diese Bäume metamathematisch definiert wird, (1) wann ein Baum eine "korrekte Definition" darstellt, und (2) wann zwei Bäume, die beide korrekte Definitionen darstellen, dasselbe Objekt definieren. (2) liefert eine Äquivalenzrelation zwischen den Bäumen—und die zugehörigen Äquivalenzklassen werden als "Objekte" des Systems eingeführt. Für diese Objekte werden anschließend—wiederum metamathematisch—die Relationen $=$, $\neq$, $\in$, $\notin$, $\swarrow$, $\searrow$, I, a, v, aC, vC definiert, und es wird ein Beweis des grundlegenden "Kohärenzprinzips" skizziert: ein Definitionsprädikat $P(x)$ ist genau dann eine korrekte Definition, wenn es genau ein "Objekt" gibt, das $P(x)$ erfüllt, und zwar wird $P(x)$ von der zu $P(x)$ gehörigen Äquivalenzklasse von Bäumen erfüllt.

Die so gewonnenen "Objekte" bilden die 0te Kategorie. Im Gegensatz zur verzweigten Typentheorie—mit welcher der Definitionismus noch am meisten Ähnlichkeit hat—gehören zur 0ten Kategorie schon Mengen beliebig hoher Stufe, z.B. in SK alle endlichen Mengen $\{0, \{0\}, \{\{0\}\}, \cdots\}$.

Teil II beginnt mit der Konstruktion höherer Kategorien. Es wird eine Metasprache über die bisherigen Bäume eingeführt, die außer den schon definierten Relationen $=$, $\in$, $\cdots$ auch Prädikate zur Beschreibung der formalen Struktur der Bäume enthält, z.B. "X ist ein Teil des Zeichens Y", "Die Anzahl der Zeichen der Gestalt X in Y ist Z", $\cdots$. Dazu benutzt Verf. neben den bisherigen Bäumen, die jetzt—zur Veranschaulichung—als einfarbig (schwarz) vorgestellt werden, weitere Bäume, deren Strecken und Knoten teilweise anders (rot) gefärbt sind. Diese zweifarbigen Bäume stellen eindeutig die Teile der einfarbigen Bäume dar. Außerdem werden noch Bäume (derselben Gestalt wie bisher) in einer dritten Farbe (blau) betrachtet, die anstelle der Individualnamen der einfarbigen Bäume beliebige zweifarbige Bäume enthalten.

Die mehrfarbigen Bäume sind jedoch nur Hilfskonstruktionen. Es werden die Prädikate über die (einfarbigen) Bäume so ausgewählt, daß sie zugleich Prädikate über die dargestellten "Objekte" sind, also verträglich mit der Äquivalenzrelation sind. Zur Zusammensetzung der Prädikate werden die Funktoren &, v, $^-$ uneingeschränkt, die Quantoren nur mit Einschränkungen zugelassen. Die Prädikate dieser Metasprache stellen zugleich wieder "Objekte" (Eigenschaften) dar. Um die Prädikate auf diese "Objekte" auszudehnen, läßt Verf. nur die beiden Möglichkeiten zu: sie sollen für alle Objekte, für die sie ursprünglich nicht definiert sind, gleichmäßig wahr oder gleichmäßig falsch sein. Mit den so erhaltenen Metaprädikaten als primitiven Prädikaten—neben $=$, $\neq$, $\in$, $\notin$, a, v, aC, vC—werden dann neue Definitionsprädikate durch Bäume konstruiert. Die beiden Prozesse der Konstruktion von Definitionsprädikaten und Metaprädikaten werden abwechselnd beliebig oft hintereinander angewendet. Die induktive Definition der Korrektheit der dabei entstehenden Definitionsprädikate wird vom Verf. nur "très approximative" (S. 67) angegeben. Ref. konnte daher den Einzelheiten nicht mehr folgen. Es ist aber plausibel, daß—wie der Verf. auf S. 99 bemerkt—die Analysis, die auf der Basis der definitionistischen SK aufgebaut werden kann, "beaucoup de ressemblance" mit der operativen Analysis des Ref. hat.

Verf. skizziert dann nicht-kroneckersche Systeme, von denen noch das "système borélien" hervorzuheben ist, bei dem als aktuale Eigenschaften nur abzählbare Mengen auftreten. Die höheren definitionistischen Systeme dürften der sog. höheren Prädikatenlogik (2. Stufe, 3. Stufe, $\cdots$) gleichwertig sein.

Ausführlicher wird das Verhältnis der definitionistischen Systeme zur üblichen Axiomatik erörtert. Man kann zunächst definieren, was ein Modell eines Axiomensystems in einem definitionistischen System ist. Die Eigenschaft, ein Modell zu sein, tritt dann an die Stelle des Axiomensystems. Wie in der Semantik kann "A ist eine logische Folgerung aus den Axiomen $A_1, \cdots, A_n$" definiert werden durch die Äquivalenz der Eigenschaften, ein Modell von $A_1, \cdots, A_n$ bzw. von $A_1, \cdots, A_n$ und A zu sein.

Zum Schluß wird die Lösung der Paradoxien vom Standpunkt des Definitionismus diskutiert. Die Russellsche Paradoxie tritt nicht auf, weil $x \notin x$ nicht als Definitionsprädikat vorkommen kann: das wird durch die Konstruktion der Bäume, die stets "stratifizierbar" sind, ausgeschlossen. Tritt $x \notin x$ als Metaprädikat auf, so kann es zwar auf sich selbst angewendet werden, ist dort aber per definitionem gleichmäßig wahr oder gleichmäßig falsch.

Die Behandlung der übrigen Paradoxien (Burali-Forti, Berry, Richard, Grelling-Weyl) ist nicht ganz so einfach.

Ein zusammenfassendes Urteil über den Definitionismus möchte der Ref. auf Grund der bisherigen Skizzen nicht fällen. Vom konstruktiven Standpunkt ist—auch bei Beschränkung auf das "système kroneckérien"—bedenklich, daß uneingeschränkt die klassische Logik verwendet wird. Für den Nicht-Konstruktivisten sollte der Definitionismus jedoch eine interessante Bereicherung der formalen Systeme sein. *P. Lorenzen* (Kiel)

4912:

★**Bernays, Paul. Axiomatic set theory. With a historical introduction by A. A. Fraenkel.** Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1958. viii+226 pp. 22.50 guilders; \$6.00.

Dies ist, was seinen Hauptteil betrifft, ein hochwissenschaftliches, abgeklärt-durchsichtig geschriebenes Buch,

das sich zum Ziele gesteckt hat, zu zeigen, wie die gesamte klassische Analyse auf der Grundlage einer formalen axiomatischen Mengenlehre entwickelt werden kann. Das im vorliegenden Buch dargestellte formale System ist eine Modifikation des in einer Reihe von Aufsätzen dargestellten, im J. Symb. Logic veröffentlichten, Systems. Die formale Entwicklung ist in einer ziemlich ins Detail gehenden Weise durchgeführt, und nur in den Anwendungen auf die Analyse, Arithmetik der Kardinalzahlen und abstrakte algebraische Theorien hat sich Verf. auf einige methodische Hinweise beschränkt, die aber durchaus genügen, um die Möglichkeit auch ihrer Einbeziehung erscheinen zu lassen.

Die Einstellung des Verfassers könnte als die der "Panmengenlehre" bezeichnet werden. In der Tat sind bei ihm z.B. die Ordinalzahlen, Kardinalzahlen, Funktionen, Folgen, natürlichen Zahlen, alles 'Mengen'. Sogar die Klassen werden bei ihm neben den Mengen nicht als ein Dualismus einer zweisortigen axiomatischen Theorie, sondern vielmehr als ein Mittel zur Einbeziehung eines Teiles von metatheoretischen Begriffen in das System selbst, eingeführt.

Der Verfasser sucht nicht den bequemsten axiomatischen Zugang zur Mengenlehre, sondern den am meisten effektiven. Er führt zunächst die beiden Gleichheits- und Extensionalitätsaxiome (E1) $a=b.\rightarrow.a\in A\rightarrow b\in A$ und (E2) $(\forall x)(x\in a\leftrightarrow x\in b)\rightarrow a=b$ ein und dann die drei Axiome der allgemeinen Mengenlehre:

(A1) $a\notin 0;$

(A2) $a\in b;c\leftrightarrow a\in b \vee a=c;$

(A3) $a\in \sum_{\mathfrak{x}}(m, \mathrm{t}(\mathfrak{x}))\leftrightarrow(\exists\mathfrak{x})(\mathfrak{x}\in m \,\&\, a\in \mathrm{t}(\mathfrak{x})),$

"0 ist eine leere Menge", "$b;c$ ist ein Menge, deren Elemente die Elemente von b und die Menge c sind", und "$\sum_{\mathfrak{x}}(m, \mathrm{t}(\mathfrak{x}))$ ist eine Menge, die die Vereinigung der Mengen $\mathrm{t}(c)$ mit $c\in m$ ist". Die drei hiermit eingeführten Konstanten sind parallel zu dem Anfangselement und dem Nachfolgeoperator der gewöhnlichen Arithmetik und zu dem Grenzprozessoperator der Theorie Cantors von den transfiniten Zahlen. Durch diese drei Axiome werden die vier Axiome: Zermelos Aussonderungsaxiom, das Paarungsaxiom, das Vereinigungsaxiom und das Fraenkelsche Ersetzungsaxiom zu beweisbaren Sätzen. Es werden sodann die "vervollständigenden" Axiome, nämlich das Potenzmengenaxiom, das Auswahlaxiom und das Unendlichkeitsaxiom eingeführt.

Kurze Inhaltsangabe des Buches: Im ersten Teil gibt Professor Fraenkel eine historische Einführung in die ursprüngliche Zermelo-Fraenkelsche Form der mengentheoretischen Axiomatik und ihre direkten Fortführungen. Im ersten Kapitel des zweiten Teiles werden der logistische Rahmen der nachfolgenden Theorie sowie der Klassenformalismus besprochen. Im zweiten Kapitel wird der Beginn der allgemeinen Mengenlehre entwickelt. Im Kapitel III wird die Theorie der Ordinalzahlen (ohne sich auf die Theorie der Ordnung zu stützen) eingeführt. Die Ordinalzahlen werden durch die Relation $\in$ wohlgeordnet. Das Kapitel IV bringt die Theorie der transfiniten Rekursion und das Kapitel V die Theorie der Mächtigkeit, Ordnung und Wohlordnung. Im Kapitel VI werden die oben erwähnten vervollständigenden Axiome eingeführt. Das Kapitel VII ist den Anwendungen des vollen Systems der Axiome (A1)–(A6) auf die klassische Mathematik gewidmet. Es werden die Theorie der reellen Zahlen, die Arithmetik der Ordinal- und Kardinalzahlen, die Gruppentheorie, die Topologie der Räume und die algebraische Theorie der Ringe in wechselndem Umfang entwickelt. Zum Schluss wird eine Verstärkung des Auswahlaxioms und das Fundierungsaxiom besprochen und eine eindeutige Korrespondenz zwischen der Klasse aller Ordinalzahlen und der Klasse aller Mengen hergestellt.

Diese Monographie ist als der erste Band eines zweibändigen Werkes gedacht, dessen zweiter Band den weiteren axiomatischen Fragen der Eliminierbarkeit, relativen Widerspruchsfreiheit und Unabhängigkeit gewidmet sein wird.

B. Germansky (Berlin)

4913:

Turquette, Atwell, R. Simplified axioms for many-valued quantification theory. J. Symb. Logic **23** (1958), 139–148.

The author shows that the Rosser-Turquette formalisation [*Many-valued logics*, North-Holland Publ. Co., Amsterdam, 1951; MR **14**, 526], of certain m-valued predicate calculi may be simplified. The previous formalisation used ten axiom schemes and two primitive rules of procedure and the new formalisation uses nine axiom schemes with modus ponens as the only primitive rule of procedure. The new method is applicable to every m-valued predicate calculus to which the previous method was applicable. It is also applicable to certain other predicate calculi and it is therefore impossible, in certain cases, to derive all the original axiom schemes in the new formalisation.

A. Rose (Nottingham)

4914:

Margaris, Angelo. A problem of Rosser and Turquette. J. Symb. Logic **23** (1958), 271–279.

The author shows that if s, t, m are integers such that $1\leq s<t<m$ then there exist a formalisation of an m-valued propositional calculus and formulae $P_{s+1}, \cdots, P_t; Q_{s+1}, \cdots, Q_t$ with the following properties. (i) Every formula which takes only truth-values belonging to the set $\{1, \cdots, s\}$ is provable. (ii) Every formula which does not take only truth-values belonging to the set $\{1, \cdots, t\}$ is unprovable. (iii) P_k and Q_k take only truth-values belonging to the set $\{1, \cdots, k\}$ but neither formula takes only truth-values belonging to the set

$$\{1, \cdots, k-1\}\quad (k=s+1, \cdots, t).$$

(iv) $P_{s+1}, \cdots, P_t$ are provable in the formalisation and $Q_{s+1}, \cdots, Q_t$ are unprovable.

The system used in the proof is not functionally complete. The corresponding problem for functionally complete systems is discussed, but the solution has not been obtained.

A. Rose (Nottingham)

4915:

★**Hermes, Hans. Zum Inversionsprinzip der operativen Logik.** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 62–68. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii+297 pp. $8.00.

Verf. gibt eine neue Formulierung des Inversionsprinzips, da die Anwendungen des Prinzips in der operativen Mathematik durch die bisherige Formulierung

[Lorenzen, *Einführung in die operative Logik und Mathematik*, Springer, Berlin-Göttingen-Heidelberg, 1955; MR **17**, 223] nicht gedeckt wurden. *P. Lorenzen* (Kiel)

4916:

★**Péter, Rózsa. Rekursivität und Konstruktivität.** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 226–233. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii + 297 pp. $8.00.

The author asks 'Can general recursive functions really be called effectively calculable, or constructive?' The first source of doubt on which the question is raised is the existence of apparent vicious circles in alternative definitions of 'general recursive', some arising from the assertion of existence of either a computation procedure or a number, others arising in connection with ordinal numbers and the attempt to extend narrower classes of recursion into the transfinite. Alternate methods of explicating the notion of 'general recursion' by extensions such as these are suggested and rejected. The conclusion is that it seems that the concept of constructivity cannot be formulated without circularity. *E. J. Cogan* (Bronxville, N.Y.)

4917:

Huzino, Seiiti. On the existence of Sheffer stroke class in the sequential machines. Mem. Fac. Sci. Kyushu Univ. Ser. A **13** (1959), 53–68.

Given two machines M_1 and M_2, the author defines a third machine $M_1 \wedge M_2$. He then defines a class of machines called the Sheffer stroke class. He then shows that any machine can be represented by finite iterations (under $\wedge$) of machines in the Sheffer stroke class.

The notation was too complicated for the reviewer to understand the definitions and proofs.

S. Ginsburg (Gardena, Calif.)

4918:

Huzino, Seiiti. Some properties of convolution machines and σ-composite machines. Mem. Fac. Sci. Kyushu Univ. Ser. A **13** (1959), 69–83.

The present paper continues the study of sequential machines by the author [see same Mem. **12** (1958), 136–158, 159–179; MR **20** #7599, #7600]. Given two machines M_1 and M_2 the author defines two machines, both depending on M_1 and M_2, called the convolution machine and the σ-composite machine. Questions about length of experiments, distinguishability of states, and strong connectedness of the new machines are studied with respect to the corresponding properties in M_1 and M_2.

S. Ginsburg (Gardena, Calif.)

4919:

★**Tarski, Alfred. What is elementary geometry?** The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 16–29. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi + 488 pp. $12.00.

An elegant axiomatic first-order system E_2 for elementary two-dimensional Euclidean geometry is presented. The variables range over points and the only non-logical predicates are the ternary predicate $\beta(x, y, z)$ meaning "y is between x and z", and the quaternary predicate $\delta(xyuv)$, meaning "x is as distant from y as u is from v". An axiom schema provides the infinite collection of elementary continuity axioms. (There are no set variables; hence, the continuity axiom cannot be given in its full strength.) The axioms fixing the dimension as 2 can be replaced by axioms making any other positive integer the dimension. Proofs of the following theorems are sketched. The models of E_2 are precisely the two-dimensional Cartesian spaces $C_2(F)$ with coordinates in a real-closed field F (Representation Theorem). A sentence in E_2 is a theorem if and only if it holds in $C_2(R)$ where R is the field of real numbers. Hence E_2 is complete and decidable. E_2 is not finitely axiomatizable. (The principal tools in these proofs are Tarski's decision method for elementary algebra and Hilbert's "theory of proportions".) If one adds to E_2 new variables, assumed to range over finite sets of points, and a membership relation $\in$ between points and finite sets of points, this new theory E_2' turns out to be undecidable (since Peano's arithmetic is relatively interpretable within it), but the problems of representation, completeness and finite axiomatizability are unsolved for E_2'. The author also considers the finitely axiomatizable subsystem E_2'' of E_2 obtained by replacing the schema of elementary continuity axioms by the axiom:

$$\bigwedge xyzx'z'u \bigvee y'[\delta(uxux') \wedge \delta(uzuz') \wedge \beta(uxz) \wedge \beta(xyz) \rightarrow \delta(uyuy') \wedge \beta(x'y'z')].$$

This, roughly speaking, means that a segment which joins two points, one inside and one outside a given circle, always intersects the circle. The models of E_2'' are precisely the Cartesian spaces $C_2(F)$, where F is a Euclidean field, i.e., an ordered field in which every positive element has a square root. It follows that E_2'' is incomplete. The author conjectures that E_2'' and every other finitely axiomatizable subtheory of E_2 is undecidable, but the question is still unresolved.

E. Mendelson (New York, N.Y.)

4920:

★**Scott, Dana. Dimension in elementary Euclidean geometry.** The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 53–67. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi + 488 pp. $12.00.

A Euclidean space is a real vector space with an inner product. An n-ary geometric relation is a function R assigning to each Euclidean space V a subset R_V of V^n such that these relations R_V are preserved under any isometry of one space into another. Generalizing results of Tarski and Vaught, the author proves a theorem concerning a new notion, "arithmetical extension of finite degree", from which he obtains the theorem: If R is a geometric relation and φ is a sentence of the first-order system with a predicate corresponding to R as the only non-logical predicate, and if φ has $m+1$ distinct variables, then φ is true in all relational systems $\langle V, R_V\rangle$, where V is a Euclidean space of dimension at least m, if and only if

φ is true in at least one such relational system. From this theorem are drawn various results concerning Tarski's first-order system of elementary Euclidean geometry with predicates for betweenness and equidistance (as given in Tarski's paper reviewed above). Let E_m be the class of sentences true for spaces of dimension m. E_m is complete, and by results of Tarski, axiomatizable and decidable. Let $E=\bigcap_{m<\omega} E_m$ be the class of sentences true in all finite dimensions. Let $E_\infty=\bigcup_{m<\omega}\bigcap_{n>m} E_n$ be the class of all sentences true in all but a finite number of dimensions. Then the only finite complete extensions of E are the theories E_m with $m<\omega$. E_∞ is the set of sentences true in all infinite-dimensional spaces and is the unique complete infinite extension of E. Both E and E_∞ are decidable.

E. Mendelson (New York, N.Y.)

SET THEORY

See also 5001.

4921:

Kondô, Motokiti. Sur la théorie projective des ensembles. C. R. Acad. Sci. Paris **248** (1959), 2940–2942.

Suite des Notes précédentes de l'Auteur [mêmes C. R. **242** (1956), 1841–1843, 1945–1948, 2084–2087, 2209–2212, 2275–2278; MR **17**, 933; **18**, 2] et passage à un niveau plus élevé. Soient S un ensemble non vide et $\rho(x)$, pour chaque $x\in S$, un nombre ordinal; soit $R^{(\alpha)}$ l'ensemble des $\rho^{-1}\alpha$. On suppose: (A.1) S est l'union des $R^{(\alpha)}$ $(\alpha<\eta)$, η étant un ordinal fixe; (A.2) $R^{(0)}=$l'ensemble des entiers; (A.3) $R^{(\alpha)}$ est un anneau commutatif rel. à $+$ et $\cdot$ et à une unité $e^{(\alpha)}$. Le segment α de S est défini comme la réunion $S^{(\alpha)}$ des $R^{(\beta)}$ $(\beta<\alpha)$. On postule l'existence de produits intérieurs (a, b) d'éléments a, b de $S^{(\alpha)}$ par des postulats suivants: (B.1) $(a, b)\in R^{(0)}$; (B.2) $(a, b\pm c)=(a, b)\pm(a, c)$; (B.3) $(a, bc)=(a, b)(a, c)$; (B.4) $(a, e^{(\alpha)})=1$ $(a\in S^{(\alpha)})$; (B.5) $b, c\in R^{(\alpha)}$, $(a, b)=(a, c)$ $(a\in S^{(\alpha)})\Rightarrow b=c$. S est dit un système fondamental de type η et $R^{(\alpha)}$ est le composant de type α de S ou un hyperdomaine. On définit en particulier le système absolu $I^{(\eta)}$ de type η en posant $I^{(0)}=R^{(0)}$, $I^{(\alpha)}$ étant l'ensemble des transformations de $\bigcup_{\beta<\alpha} I^{(\beta)}\to I^{(0)}$. On définit une topologie sur S en se servant de la multiplication postulée et d'une famille d'ensembles $\subseteq S$. Cela induit une topologie dans la somme directe de chaque suite finie $(R^{(\alpha_1)}, \cdots, R^{(\alpha_n)})$ de type $\xi=\max\{\alpha_1, \cdots, \alpha_n\}$. En considérant les G et F de ces sommes directes et leurs projections et complémentations on arrive à des ensembles hyperprojectifs de type ξ. En partant d'un $S_0\subseteq S$, on considère la théorie $T(S_0, S)$ des ensembles sur S rel. S_0; elle est dite du type égal à celui de S.

Đ. Kurepa (Princeton, N.J.)

4922:

Kondô, Motokiti. Sur la nommabilité d'ensembles de type supérieur. C. R. Acad. Sci. Paris **248** (1959), 3099–3101.

En partant d'un $S_0\subset S$ et de la théorie $T(S_0, S)$ [voir la Note précédente], on construit tout d'abord des hyperpolynomes (S_0, S) d'une façon récurrente (chaque entier, $x^{(\alpha_j)}$ $(\alpha_j\neq 0)$, $(x^{(\alpha_i)}, x^{(\alpha_j)})$, etc., en sont des exemples). On passe alors à la définition des ensembles hyperélémentaires, à la projection et complémentation de ceux-ci, etc. On établit une hiérarchie parmi ces ensembles et conjointement une hiérarchie hyper-projective des formules de la théorie $T(S_0, S)$.

Đ. Kurepa (Princeton, N.J.)

COMBINATORIAL ANALYSIS

See also 4987, 5200, 5201, 5202, 5230.

4923:

Seidenberg, A. A simple proof of a theorem of Erdös and Szekeres. J. London Math. Soc. **34** (1959), 352.

The theorem [Compositio Math. **2** (1935), 463–470] asserts that if in a sequence of more than mn distinct real numbers the longest decreasing subsequence has at most m terms, then some increasing subsequence has more than n terms.

4924:

Gould, H. W. Note on a paper of Steinberg. Math. Mag. **33** (1959/60), 46–48.

The paper referred to [same Mag. **31** (1957/58), 207–209] was listed in MR **20** #4503.

4925:

Lohne, Johs. Power sums of natural numbers. Nordisk Mat. Tidskr. **6** (1958), 155–158. (Norwegian)

The power sums

$$S_r(m)=\sum_{n=1}^{m} n^{2r+1} \quad\text{and}\quad T_r(m)\sum_{n=1}^{m} n^{2r+2}$$

are considered. It is shown that

$$S_r(m)=\sum_{s=1}^{r+1}\frac{K_{rs}}{2s}\frac{(m+s)!}{(m-s)!},$$

$$T_r(m)=\frac{2m+1}{2}\sum_{s=1}^{r+1}\frac{K_{rs}}{2s+1}\frac{(m+s)!}{(m-s)!},$$

where the coefficients K_{rs} are the same in both formulas. These coefficients are positive integers, and they form a numerical triangle specified by $K_{r,1}=K_{r,r+1}=1$ and $K_{r+1,s}=K_{r,s-1}+s^2K_{rs}$. In comparison to the well-known expressions for S_r and T_r involving the Bernoulli numbers [cf. C. Jordan's *Calculus of finite differences*, Chelsea, New York, 1950; MR **1**, 74 (1939 ed.); p. 130], the author's formulas have integral rather than fractional coefficients, and involve significantly fewer non-zero terms for $m<r$.

S. W. Golomb (Pasadena, Calif.)

4926:

Tambs Lyche, R. Supplement to the preceding article. Nordisk Mat. Tidskr. **6** (1958), 159–161. (Norwegian)

Letting K_{rs} denote the coefficients in Lohne's power sum formulas [see preceding review], it is shown that

$$K_{rs}=\frac{1}{(2s-1)!}\sum_{t=2}^{s} A_{st}(t^{2r+1}-t),$$

where in turn

$$A_{st}=(-1)^{s+t}\frac{t}{s}\binom{2s}{s-t}.$$

The explicit expression

$$K_{rs} = \sum_{t=2}^{s} (-1)^{s+t} \frac{2t^2(t^{2r}-1)}{(s-t)!(s+t)!}, \quad s \geqq 2,$$

is derived, and from it Lohne's recursion relation

$$K_{r+1,\,s} = K_{r,\,s-1} + s^2 K_{rs}$$

is deduced. This makes extensive use of generating function techniques, and is thereby less elementary than Lohne's article. *S. W. Golomb* (Pasadena, Calif.)

ORDER, LATTICES

4927:

Morel, Anne C. On the arithmetic of order types. Trans. Amer. Math. Soc. **92** (1959), 48–71.

Let Greek letters denote order types. If $\gamma\alpha=\gamma\beta$ implies $\alpha=\beta$ for every α and β, then γ is called a left cancelling type. I. (a) The following three assertions are equivalent: γ is a left cancelling type; for every α, $\gamma\neq\gamma+\gamma\alpha+\gamma$; for every $\alpha\neq 1$, $\gamma\neq\gamma\alpha$. (b) Every scattered type is a left cancelling type. (c) Every nonzero, gapless, unbordered type is a left cancelling type. II. (a) If $\beta^2=\gamma^2$ and β is gapless and unbordered, then $\beta=\gamma$. (b) If $\alpha 2$ is not a subtype of α (in particular, if α is scattered), then the equation $\xi^n=\alpha$, where n is a natural number, has at most one solution. (c) For every integer $n\geqq 2$ and every cardinal number $\mathfrak{m}\in\{0, 1, 2, \cdots, \aleph_0, 2^{\aleph_0}\}$, there exist $2^{\aleph_0}$ enumerable, nonscattered types α such that the equation $\xi^n=\alpha$ has exactly $\mathfrak{m}$ solutions. *F. Bagemihl* (South Bend, Ind.)

4928:

Choe, Tae Ho. The topologies of partially ordered set with finite width. Kyungpook Math. J. **2** (1959), 17–22.

The subject is partially ordered sets in which (*) each subset A which is not closed in the interval topology contains a chain which has a supremum or infimum not in A. It is shown that if (**) the cardinal numbers of sets of incomparable elements have a finite upper bound, then (*); and it is asserted that for a lattice with O and I, (*) is equivalent to the non-existence of infinite sets of incomparable elements. The first result follows at once from Dilworth's theorem that if (**), then the set is a finite union of chains [Ann. of Math. (2) **51** (1950), 161–166; MR **11**, 309]. For a counterexample to the second, take the sublattice of $\eta\times\eta$ consisting of all (x, y) with each of x and y either ± 1 or of the form $\pm(1-1/n)$, except $(1, -1)$ and $(-1, 1)$. *J. Isbell* (Lafayette, Ind.)

4929:

Kalman, J. A. On the postulates for lattices. Math. Ann. **137** (1959), 362–370.

Let $\mathfrak{L}$ be the family of all algebraic systems $\mathfrak{l}$ with two binary operations $\wedge$ and $\vee$. Let ω be the following set of laws for such systems: $x\wedge x=x$, $x\wedge y=y\wedge x$, $x\wedge(y\wedge z)=(x\wedge y)\wedge z$, $x\wedge(x\vee y)=x$, $(x\vee y)\wedge x=x$, $x\wedge(y\vee x)=x$, $(y\vee x)\wedge x=x$, and the laws obtained by interchanging $\wedge$ and $\vee$. For each $\xi\subseteq\omega$ let $\mathfrak{L}\xi$ be the family of all $\mathfrak{l}$ in $\mathfrak{L}$ which obey all the laws in ξ. The following problems are considered: "P_ξ: to find all the subsets of ω which constitute an independent system of axioms for the family $\mathfrak{L}\xi$; and Q_ξ: to find all the laws in ω which are obeyed by every $\mathfrak{l}$ in $\mathfrak{L}\xi$." The author gives tables from which one can fairly easily determine the solution of the problem P_ξ or Q_ξ for any $\xi\subseteq\omega$. It is shown that all relations among the laws in ω are formal consequences of nine indicated relations. A set of (twelve) examples is given which essentially comprises a collection of counterexamples which are together adequate for the testing of any conjectured relation among the laws in ω.

As the author observes, the problem P_ω has been solved by Sorkin [Ukrain. Mat. Ž. **3** (1951), 85–97; MR **14**, 612]. Some applications to P. Jordan's studies [Abh. Math. Sem. Univ. Hamburg **21** (1957), 127–138; MR **19**, 524] of noncommutative lattices are pointed out.

M. Kolibiar (Bratislava)

4930:

Learner, A. Hilbert's function in a semi-lattice. Proc. Cambridge Philos. Soc. **55** (1959), 239–243.

Let L be a partially ordered set with a greatest element Q. We assume: (1) Given A, $B\in L$ such that $A\leq B$, there exists a saturated chain between A and B, and two such chains have the same length, denoted by $l(B/A)$; (2) any two elements A, $B\in L$ have a l.u.b., denoted by $A+B$; (3) a commutative associative product AB is defined in L, satisfies $A(B+C)=AB+AC$, $AQ=A$, and is compatible with the order relation; (4) if $A'<A$, $B'<B$, $l(A/A')=1$ and $l(B/B')=1$, then $l((A+X)/(A'+X))\leq 1$ and $l(AB/(AB'+A'B))\leq 1$. Using the combinatorial method of Sperner's for Hilbert theorem on characteristic polynomials [Abh. Math. Sem. Univ. Hamburg **7** (1930), 149–163], the author proves that, for $A\in L$, $l(A^n/A^{n+1})$ is a polynomial in n for n large. This abstract result applies to many known cases (polynomial rings, semi-local rings, modules over semi-local rings). *P. Samuel* (Urbana, Ill.)

4931:

Amemiya, Ichiro; and Halperin, Israel. Coordinatization of complemented modular lattices. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 70–78.

K. Fryer and I. Halperin [same Proc. **61** (1958), 142–161; MR **20** #6374] have given a lattice-theoretic coordinate construction for a complemented modular lattice with a normalized 3-frame and satisfying a generalization of the "uniqueness of harmonic conjugate point" condition of Moufang which leads to an idempotent-associative, alternative, regular ring R. In this paper it is shown that R can actually be used to coordinatize the lattice. Indeed, the lattice is isomorphic to the lattice of all M-sets. An M-set is defined as follows: If e_1, e_2, e_3 are idempotents of R and α_{21}, α_{31}, α_{32} are such that $e_i\alpha_{ij}=\alpha_{ij}$, then an M-set shall consist of all vectors

$$\alpha=(\alpha_1, \alpha_2, \alpha_3)=\delta_1(e_1, 0, 0)+\delta_2(\alpha_{21}, e_2, 0)+\delta_3(\alpha_{31}, \alpha_{32}, e_3)$$

such that for some i, α_i is idempotent and $\alpha_j=0$ for $j>i$ while $\alpha_i\alpha_j=\alpha_j$ for $j<i$.

R. P. Dilworth (Pasadena, Calif.)

4932:

Halmos, Paul R. Free monadic algebras. Proc. Amer. Math. Soc. **10** (1959), 219–227.

A free monadic extension of a Boolean algebra B is a monadic algebra A with the properties that (i) B is a

Boolean subalgebra of A, (ii) A is (monadically) generated by B, (iii) every Boolean homomorphism of B into a monadic algebra C can be extended to a monadic homomorphism of A into C. The following steps are shown to lead to a free monadic extension A of an isomorphic image of B: Let W be the space of all functions on B into a two-element Boolean algebra 2. Let X be the subspace of $W\times W$ consisting of all (y, v) such that y is a homomorphism, v is a hemi-morphism and $y \leq v$. Let A be the dual algebra of X, i.e., the set of all continuous functions on X into 2, and for p in A define $\exists p$ by letting $(\exists p)(y, v)$ be the supremum of $p(u, v)$ ranging over all u with $(u, v)\in X$. The mapping h of B into A is defined by letting $h(p)(y, v)=p(y)$ for all $p\in B$ and $(y, v)\in X$.

B. Jónsson (Minneapolis, Minn.)

GENERAL MATHEMATICAL SYSTEMS

See also 5103.

4933:

Clarke, A. Bruce. On the representation of cardinal algebras by directed sums. Trans. Amer. Math. Soc. **91** (1959), 161–192.

This paper develops construction methods by means of which cardinal algebras (C.A.'s) subject to suitable conditions can be systematically built up from more elementary types of algebras. A C.A. $\mathfrak{A}=\langle A, +, \sum\rangle$ is said to be primary if $\infty\cdot a<\infty\cdot b$ always implies that $a<b$. If this implication holds whenever a is disjunctively indecomposable, then $\mathfrak{A}$ is said to be semi-primary. By the width of $\mathfrak{A}$, $w(\mathfrak{A})$, is understood the sup of the cardinal numbers of all sets of pairwise disjunctive non-zero elements of A. The principal results concern the class of all primary C.A.'s and the class of all semi-primary C.A.'s of finite width.

Since C.A.'s are partially ordered systems, there is a natural way of defining a directed sum of C.A.'s over an index algebra. Actually it is found to be convenient to allow some of the summands to be slightly more general algebras, namely systems obtained by removing the zero elements from C.A.'s. On the other hand the most interesting cases turn out to be those in which the index algebra is idemmultiple and simply ordered. This leads to a theorem asserting the existence of a unique representation of an arbitrary C.A. as a sum of linearly indecomposable summands over an idemmultiple simply ordered index algebra. If the given algebra is primary, then so are the summands, and it turns out that in this case the summands must be either simple or else idemmultiple.

For the study of semi-primary C.A.'s a new operation is introduced, generalizing both the direct product and the ordered sum of two C.A.'s. Given two C.A.'s, $\mathfrak{A}$ and $\mathfrak{B}$, and a homomorphism φ of $\mathfrak{B}$ into the C.A. $I(\mathfrak{A})$ of all ideals of $\mathfrak{A}$, a congruence relation R over the direct product $\mathfrak{A}\times\mathfrak{B}$ is defined by letting $\langle a, b\rangle R\langle a', b'\rangle$ if and only if $b=b'$ and $a\equiv_{\varphi(b)}a'$. The quotient $(A\times B)/R$ turns out to be a C.A., and is called the star product of $\mathfrak{A}$ and $\mathfrak{B}$ modulo φ—in symbols $\mathfrak{A} *_\varphi \mathfrak{B}$. The class of all semi-primary C.A.'s of finite width can now be described as the smallest class K having the following properties: (i) If $\mathfrak{A}$ is a primary C.A. and $w(\mathfrak{A})<\infty$, then $\mathfrak{A}\in K$. (ii) If $\mathfrak{A}\cong\mathfrak{B}\in K$, then $\mathfrak{A}\in K$; (iii) If $\mathfrak{A}$, $\mathfrak{B}\in K$, then $\mathfrak{A}\times\mathfrak{B}\in K$. (iv) If $\mathfrak{A}\in K$, $\mathfrak{B}$ is a primary C.A. and φ is a homomorphism of $\mathfrak{B}$ into $F(A)$ (the factor algebra of $\mathfrak{A}$) with $w(\text{kernel }\varphi)<\infty$, then $\mathfrak{A} *_\varphi \mathfrak{B}\in K$.

B. Jónsson (Minneapolis, Minn.)

CLASSICAL ALGEBRA

4934:

Simionescu, Gheorghe D. La réduction de l'équation générale de II-ème degré, dans le cas $\delta=0$, par une translation suivie d'une rotation. Bul. Inst. Politehn. Bucureşti **20** (1958), no. 2, 33–38. (Romanian. Russian, English, French and German summaries)

THEORY OF NUMBERS

See also 4925, 4926, 5013, 5014, 5160.

4935:

Schinzel, André. Sur les nombres composés n qui divisent a^n-a. Rend. Circ. Mat. Palermo (2) **7** (1958), 37–41.

The main result proved in this paper is the following. Given any integer $a>1$, there exist arbitrarily large primes p, q with $p\neq q$ such that $pq\mid a^{pq}-a$. The proof is elementary and straightforward. The motivation for this problem arises in trying to prove that there are infinitely many pseudo-primes of Carmichael: namely, do there exist infinitely many composite integers n such that $n\mid a^n-a$ for all a. One might remark that for n odd and square-free, n being a pseudo-prime is equivalent to saying the denominator of the Bernoulli number B_{n-1} is $2n$.

N. C. Ankeny (Cambridge, Mass.)

4936:

Schinzel A.; et Sierpiński W. Sur certaines hypothèses concernant les nombres premiers. Acta Arith. **4** (1958), 185–208; erratum **5** (1959), 259.

The authors derive many consequences of "Hypothesis H": if $f_1(x)$, $f_2(x)$, $\cdots$, $f_r(x)$ are integral-valued polynomials, $\prod_j f_j(a)\not\equiv 0 \pmod p$ for any prime p and some a, then there exist infinitely many integers n for which $f_1(n)$, $f_2(n)$, $\cdots$, $f_r(n)$ are all primes. Examples of such consequences: (1) There are infinitely many pseudo-primes [see preceding review]. (2) If a is a square-free integer and $|a|>1$, then there exist infinitely many primes for which a is a primitive root.

Many of these consequences are much weaker than the "H Hypothesis". The "H Hypothesis" would have many interesting consequences in the structure of finite groups which are not mentioned here.

N. C. Ankeny (Cambridge, Mass.)

4937:

Schinzel, A. Sur un problème concernant le nombre de diviseurs d'un nombre naturel. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 165–167.

It has been proved by A. Wintner [*The theory of measure in arithmetical semi-groups*, Baltimore, Md., 1944; MR **7**, 367, p. 14] and more recently in a different way by S. Golomb [Nordisk Mat. Tidskr. **4** (1956), 24–29; MR **17**,

944] that for every integer $k>1$ there is an integer c such that $\mu(c+1)=\mu(c+2)=\cdots=\mu(c+k)$, where $\mu(n)$ is the Möbius function. It is not known whether an analogous theorem is true for some of the other familiar numerical functions, in particular for $\theta(n)$, the number of divisors of n. It is shown in this paper that from a conjecture made by the author and W. Sierpiński ["Hypothesis H", preceding review] it would follow that there are infinitely many integers c for each k such that $\theta(c+1)=\theta(c+2)=\cdots=\theta(c+k)$. *R. J. Levit* (San Francisco, Calif.)

4938:

Brandt, Heinrich; und Intrau, Oskar. Tabellen reduzierter positiver ternärer quadratischer Formen. Abh. Sächs. Akad. Wiss. Math.-Nat. Kl. **45** (1958), no. 4, 261 pp. DM 16.

The authors use the notation

$$f = a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_2x_3+a_5x_3x_1+a_6x_1x_2$$

for ternary quadratic forms, and

$$d = a_1a_4^2+a_2a_5^2+a_3a_6^2-a_4a_5a_6-4a_1a_2a_3$$

is the formula for the discriminant. A form in which the a's are integers is called "primitive" if 1 is the g.c.d. of the a's. This table lists all reduced primitive positive ternary quadratic forms with integral coefficients with discriminants from -2 to -1000. There are over 36,000 forms listed. [Cf. the shorter tables of the reviewer, Nat. Res. Council Bull. no. 97 (1935)].

Two forms are of the same genus ("verwandt") if one may be taken into the other by a non-singular linear transformation with rational coefficients. The fundamental discriminant ("Stammdiskriminante") of a genus is the least discriminant among the forms of the genus with integral coefficients.

The adjugate form of f has the coefficients

$$a_4^2-4a_2a_3,\quad a_5^2-4a_3a_1,\quad a_6^2-4a_1a_2,$$
$$4a_1a_4-2a_5a_6,\quad 4a_2a_5-2a_4a_6,\quad 4a_3a_6-2a_4a_5.$$

The author denotes by I_1 the g.c.d. of these coefficients and defines I_2 by $I_1^2I_2=16d$. Two forms with the same invariants I_1, I_2 are said to be of the same order and I is defined by $I=I_1I_2/16$.

The basic conditions for a reduced form are

$$0<a_1\leqq a_2\leqq a_3,\quad |a_6|\leqq a_1,\quad |a_5|\leqq a_1,\quad |a_4|\leqq a_2,$$

and, in case a_4, a_5, a_6 are all negative,

$$|a_4+a_5+a_6|\leqq a_1+a_2.$$

These do not define in all cases a unique reduced form and the author merely sketches further considerations leading to unicity. He is not aware of or chooses to disregard the complete conditions obtained laboriously by L. E. Dickson [*Studies in the theory of numbers*, Chicago, 1930, Chap. IV].

In the table, forms for each discriminant are classified according to order and genus and the following invariants given: the number of automorphs, the number of forms in each genus, the prime factors of the discriminant, I_1, I_2, I and the related invariants of Minkowski, the fundamental discriminant and the characters.

This is a monumental piece of work and should be of great service to those working with quadratic forms.
B. W. Jones (Mayaguez, Puerto Rico)

4939:

Korobov, N. M. Estimates of trigonometric sums and their applications. Uspehi Mat. Nauk **13** (1958), no. 4 (82), 185–192. (Russian)

Let $f(x)=\alpha_1x+\alpha_2x^2+\cdots+\alpha_{n+1}x^{n+1}$ be a polynomial with real coefficients. For $\nu=s+2, s+3, \cdots, 3s$, where $1\leqq s\leqq(n+1)/3$, the coefficients α_ν are assumed to be rational and $\alpha_\nu=a_\nu/q$, where a_ν and q are integers. The s-rowed determinant

$$\left|\binom{s+i+j}{j}a_{s+i+j}\right|\quad (i, j = 1, 2, \cdots, s)$$

is denoted by Δ_s. Let δ be a fixed positive number less than 1/3 and suppose that $n\delta\leqq s\leqq(n+1)/3$, $s+1\leqq r\leqq 2s(1-\delta)$, $q=P^r$ and $(\Delta_s, q)=1$, where P and r are integers. The author proves that there exist positive constants $C=C(\delta)$ and $\gamma=\gamma(\delta)$ such that

$$\left|\sum_{x=1}^{P} e^{2\pi i f(x)}\right| < CP^{1-\gamma/n^2}.$$

This represents an improvement of results announced by him in three earlier papers [Dokl. Akad. Nauk SSSR **118** (1958), 231–232, 431–432; **119** (1958), 433–434; MR **20** #6393, #6394, #6395]. The improvement consists mainly in the fact that only some of the coefficients need be rational. This enables him to approximate to $(1+x/Q)^{it}$ by a logarithmic series only some of whose coefficients are rational and have denominator q satisfying $(\Delta_s, q)=1$. In this way he obtains the new estimate

$$\zeta(1+it) = O(\log^{2/3}|t|),$$

a result which he states has been obtained independently by I. M. Vinogradov, but not yet published by him.

Corresponding improvements can be obtained in other number-theoretic problems. Thus he obtains the estimates

$$\pi(x) = \mathrm{li}\, x+O\{x\exp(-a\log^{3/5}x)\}$$

and

$$\sum_{n\leqq x}\sigma(n) = \frac{\pi^2}{12}x^2+O\{x(\log x)^{2/3+\varepsilon}\},$$

where $\sigma(n)$ is the sum of the divisors of n.

The proof of the estimate for the trigonometric sum depends on three lemmas which are too complicated to quote here; one of them is an application of a mean-value theorem of Vinogradov. *R. A. Rankin* (Glasgow)

4940:

Mironov, V. T. On the zeros of the Riemann zeta function. Mat. Sb. N.S. **45(87)** (1958), 397–400. (Russian)

The author finds a precise upper bound for the real parts of the zeros, resembling his criterion for the truth of the Riemann hypothesis [Izv. Akad. Nauk SSSR Ser. Mat. **15** (1951), 91–94; MR **13**, 122]. The other result proved is that for any $\varepsilon>0$, there is a $T>0$ such that if $\zeta(z)$ has no zeros in the rectangle $h\leqq\Re z\leqq 1$, $0\leqq\Im z\leqq T$, where $\frac{1}{2}<h<1$, then it has no zeros in the half-plane $\Re z>h+\varepsilon$. While the existence of such a $T(\varepsilon)$ appears trivial, his argument could no doubt be used to obtain a suitable $T(\varepsilon)$ explicitly. *F. V. Atkinson* (Canberra)

4941:

Bochner, S. Theorems on analytic continuation which

occur in the study of Riemann's functional equation. J. Indian Math. Soc. (N.S.) **21** (1957), 127–147.

Given the sequences $0<\lambda_n\uparrow\infty$, $0<\mu\uparrow\infty$, and a real number $\delta>0$, the problem of determining the maximum number of linearly independent solutions (φ, ψ), where $\varphi(s)=\sum a_n\lambda_n^{-s}$ and $\psi(s)=\sum b_n\mu_n^{-s}$, of the functional equation

$$\pi^{-1/2s}\Gamma(\tfrac{1}{2}s)\varphi(s) = \pi^{-1/2(\delta-s)}\Gamma\{\tfrac{1}{2}(\delta-s)\}\psi(\delta-s),$$

has recently been studied by S. Bochner and K. Chandrasekharan [*] [Ann. of Math (2) **63** (1956), 336–360; MR **18**, 19] and K. Chandrasekharan and S. Mandelbrojt [**] [ibid. **66** (1957), 285–296; MR **19**, 635]. Bochner and Chandrasekharan applied a theorem of Polya [S.-B. Preuss. Akad. Wiss. **22** (1923), 45–50] on the location of singularities of an analytic function defined by a Dirichlet series $\sum a_n e^{-2\pi\lambda_n s}$ on its axis of convergence. The theorem requires two assumptions: (a) $\lim n\lambda_n^{-1}=D<\infty$; (b) $\liminf(\lambda_{n+1}-\lambda_n)>0$. Some of the results of [*] likewise required both the assumptions. However, it was later shown in [**] that the application of Polya's theorem could be avoided, and an inequality of Mandelbrojt [*Séries adhérentes, régularisation des suites, applications*, Gauthier-Villars, Paris, 1952; MR **14**, 542; Theorem 3.7, I] used instead. This made it possible to remove the restriction (b) on λ_n in the results of [*]. Subsequently, Bochner studied [Ann. of Math. (2) **67** (1958), 29–41; MR **19**, 943] functional equations with products of gamma factors in place of the single gamma factors $\Gamma(\frac{1}{2}s)$, $\Gamma(\frac{1}{2}(\delta-s))$—the products being subject to some arithmetical restrictions, still using the theorem of Polya [loc. cit.]. Here he shows that by the use of Mandelbrojt's theorem [loc. cit.] it is possible to relax restriction (b) on λ_n, even in the case of multiple gamma factors. He incidentally gives a variation on the theorem of Mandelbrojt, and discusses some lemmas on the analytic continuation of certain families of power series.

K. Chandrasekharan (Bombay)

4942:

Kahane, J. P.; et Mandelbrojt, S. Sur l'équation fonctionnelle de Riemann et la formule sommatoire de Poisson. Ann. Sci. École Norm. Sup. (3) **75** (1958), 57–80.

[* and ** below indicate the references in #4941.]

Pursuing a line of investigation initiated by Hamburger [Math. Z. **10** (1921), 240–254; **11** (1922), 224–245; **13** (1922), 283–311; Math. Ann. **85** (1922), 129–140] and continued, in a more general setting, by Bochner [Ann. of Math. (2) **53** (1951), 332–363; MR **13**, 920], Bochner and Chandrasekharan [*] and Chandrasekharan and Mandelbrojt [C. R. Acad. Sci. Paris **242** (1956), 2793–2796; MR **18**, 195 and **], the authors study functional equations modelled on, but more general than, the functional equation of Riemann's Zeta-function.

Let $\{\lambda_n\}$, $\{\mu_n\}$ $(n\geq 1)$ be two sequences of positive numbers increasing to infinity, and let $\delta>0$. Let s be a complex variable, $s=\sigma+i\tau$. Let the triplet $\{\delta, \lambda_n, \mu_n\}$ be called a 'label'. One speaks of a solution of the functional equation

$$(1)\qquad \pi^{-1/2s}\Gamma(\tfrac{1}{2}s)\varphi(s) = \pi^{-1/2(\delta-s)}\Gamma\{\tfrac{1}{2}(\delta-s)\}\psi(\delta-s)$$

with the label $\{\delta, \lambda_n, \mu_n\}$, if there exist two Dirichlet series $\varphi(s)=\sum a_n\lambda_n^{-s}$, $\psi(s)=\sum b_n\mu_n^{-s}$, and a function $\chi(s)$ which is holomorphic and uniform in a domain $|s|>R$, such that $\lim_{|\tau|\to\infty}\chi(\sigma+i\tau)=0$ uniformly in every segment $\sigma_1\leq\sigma\leq\sigma_2$, and such that for some pair of real numbers α, β, we have $\chi(s)=\pi^{-1/2s}\Gamma(\frac{1}{2}s)\varphi(s)$ for $\sigma>\alpha$, and $\chi(s)=\pi^{-1/2(\delta-s)}\Gamma\{\frac{1}{2}(\delta-s)\}\psi(\delta-s)$ for $\sigma<\beta$. The authors study the relationship between such a functional equation and a summation formula of the type

$$(2)\qquad \sum_{-\infty}^{\infty} a_n f(-\lambda_n) = \sum_{-\infty}^{\infty} b_n F(\mu_n),$$

where $f\in L_1(-\infty, \infty)$, and F is the Fourier transform of f. They refine a theorem of Hamburger [Math. Ann. **85** (1922), 129–140] by proving that with suitable restrictions on the label and on the class of functions f, one can pass from (1) to (2) or from (2) to (1). An interesting by-product is the result that under very light restrictions on the gap $(\lambda_{n+1}-\lambda_n)$, equation (1) has no solution if δ is an odd number greater than 3. The method of proof, which is both novel and elegant, combines the technique of almost-periodic (Schwartz) distributions with a formula first established in [*] (Th. 2.1), and since simplified by the reviewer, which exhibits the $\{\mu_n\}$ as the exponents of a Dirichlet series, and the $\{\lambda_n\}$ as the singularities of its sum-function on the axis of convergence. The authors also refine some of the results previously obtained in [*] and [**], on the maximum number of linearly independent solutions of (1), by replacing the notion of upper or lower density of the sequence (λ_n) by the notion of upper or lower density of repartition [J. P. Kahane, Ann. Inst. Fourier, Grenoble **7** (1957), 293–314; MR **21** #1489]. Finally they obtain a generalization of a theorem proved earlier in [**] (Th. 3),—with the aid of an interesting result of S. Agmon [Bull. Res. Council Israel **3** (1954), 385–389; MR **16**, 28]—by showing that if the upper and lower densities of repartition of the sequences (λ_n), (μ_n) exist, and equation (1) is satisfied with δ odd, then each of the sequences admits a finite base. They also obtain several properties of the sequences $(\lambda_n\pm\lambda_m)$ and $(\mu_n\pm\mu_m)$. *K. Chandrasekharan* (Bombay)

4943:

Killgrove, R. B.; and Ralston, K. E. On a conjecture concerning the primes. Math. Tables Aids Comput. **13** (1959), 121–122.

The authors consider the absolute differences, defined by $P_{ij}=|P_{i-1,\,j+1}-P_{i-1,\,j}|$, of the sequence $\{P_{0j}\}$ where P_{0j} is taken to be the jth prime number. They verify that a conjecture due to Norman L. Gilbreath (unpublished, 1958) that $P_{i0}=1$ for all $i>0$ is satisfied for $P_{0j}<792{,}722$. The calculations were carried out by the computer SWAC using a sieve of primes prepared by D. H. Lehmer.

C. B. Haselgrove (Manchester)

4944:

Wang, Yuan. On sieve methods and some of their applications. Sci. Sinica **8** (1959), 357–381.

The present paper forms the English translation of an earlier one published by the author in Chinese [see Acta Math. Sinica **8** (1958), 413–429; MR **21** #1958].

K. Mahler (Manchester)

4945:

Kubilyus, I. P. Convolutions of arithmetic functions and limit theorems for sums of independent random variables. Vestnik Leningrad. Univ. **14** (1959), no. 1, 30–33. (Russian. English summary)

Let $a_k(m)$ $(k=1, 2, \cdots)$ be a sequence of arithmetic functions with the properties (1) $a_k(m)\geq 0$ for all m and k, (2) the series $\sum_{m=1}^{\infty}a_k(m)=s_k$ are convergent and $s_k\neq 0$.

Define $A_n(m)=\sum_{m_1m_2\cdots m_n=m} a_1(m_1)a_2(m_2)\cdots a_n(m_n)$ and $\Phi_n(x)=(s_1s_2\cdots s_n)^{-1}\sum_{\ln m<x} A_n(m)$; $A_n(m)$ is called the convolution of the arithmetic functions $a_1(m), \cdots, a_n(m)$. The author puts $F_k(x)=s_k^{-1}\sum_{\ln m<x} a_k(m)$, and notes that since $F_k(x)$ and $\Phi_n(x)$ are monotonic, non-decreasing, continuous from the left, both tend to 0 as $x\to-\infty$ and to 1 as $x\to\infty$, and since $\Phi_n(x)$ is the convolution of $F_1(x), \cdots, F_n(x)$ in the function-theoretic sense, the behaviour of $\Phi_n(x)$ for large n may be described with the help of probability limit theorems for sums of independent variables. He then states three general theorems of this kind, analogues respectively of theorems due to Gnedenko and Groshev, Lindeberg and Feller, and Esseen [see Gnedenko and Kolmogorov, *Predel'nye raspredeleniya dlya summ nezavisimyh slučaĭnyh veličin*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949; translated by K. L. Chung as *Limit distributions for sums of independent random variables*, Addison-Wesley, Cambridge, Mass., 1954; MR **12**, 839; **16**, 52]. By way of illustration, it will suffice for the purpose of this review to state a special case of the author's third theorem: Let $a_1(m)=a_2(m)=\cdots=a_n(m)=m^{-2}$, when $m^2A_n(m)$ becomes $\tau_n(m)$, the number of representations of m as a product of n integers. Then

$$(6/\pi^2)^n \sum_{\ln m<x\sigma\sqrt{n}+cn} \tau_n(m)m^{-2} = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}du + \beta(1-x^2)(6\sigma^3\sqrt{(2\pi n)})^{-1}e^{-x^2/2}+O(n^{-1})$$

where c, σ and β are certain numerical constants associated with $\sum_{m=1}^{\infty} m^{-2}$. *H. Halberstam* (London)

4946:

Starčenko, L. P. Construction of sequences jointly normal with a given one. Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 757–770; erratum, **23** (1959), 635–636. (Russian)

A sequence of digits $\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots$ $(0\leq\varepsilon_n<g)$ is called normal if, for every Δ_s of s digits,

$$\lim_{P\to\infty}\frac{N_P(\Delta_s)}{P}=\frac{1}{g^s},$$

where $N_P(\Delta_s)$ is the number of times that Δ_s occurs among the first $P+s-1$ digits of the sequence. This is generalized to a system of l sequences of digits $\varepsilon_1^{(k)}, \varepsilon_2^{(k)}, \varepsilon_3^{(k)}, \cdots$ $(1\leq k\leq l,\ 0\leq\varepsilon_n^{(k)}<g_k)$ which may be regarded as forming a matrix of l rows and infinitely many columns. Let Δ_s be any matrix of l rows and s columns, the kth row consisting of non-negative integers less than g_k $(1\leq k\leq l)$, and let $N_P(\Delta_s)$ be the number of times that Δ_s occurs as a submatrix of s consecutive columns in the first $P+s-1$ columns of the infinite matrix. Then the l sequences of digits are called jointly normal if, for every Δ_s,

$$\lim_{P\to\infty}\frac{N_P(\Delta_s)}{P}=\frac{1}{(g_1g_2\cdots g_l)^s}.$$

This definition differs slightly from that given by N. M. Korobov [same Izv. **19** (1955), 361–380; MR **17**, 590]. A sufficient condition for joint normality is that

$$\limsup_{P\to\infty}\frac{N_P(\Delta_s)}{P}<\frac{c}{(g_1g_2\cdots g_l)^s}$$

for some positive constant c.

Suppose that $\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots$ is a given normal sequence with base $g\geq 2$. The author constructs an infinity of sequences $\varepsilon_1^{(k)}, \varepsilon_2^{(k)}, \varepsilon_3^{(k)}, \cdots$ $(k=1, 2, 3, \cdots)$ with the same base g such that, for any positive l, the sequences $\{\varepsilon_n\}, \{\varepsilon_n^{(1)}\}, \cdots, \{\varepsilon_n^{(l-1)}\}$ are jointly normal. A conjecture of Korobov [ibid. **14** (1950), 215–238; MR **12**, 321] that every normal sequence can be expressed as a sequence of the first digits (in the scale of g) of some completely uniformly distributed sequence is answered in the affirmative. Further, let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences of numbers in the interval $[0, 1]$ and write $Q_j^{(2s)}$ for the point with coordinates $(\alpha_j, \alpha_{j+1}, \cdots, \alpha_{j+s-1}, \beta_j, \beta_{j+1}, \cdots, \beta_{j+s-1})$ in the $2s$-dimensional unit cube. The two sequences are said to be jointly completely uniformly distributed if, for every positive integer s, the sequence of points $Q_1^{(2s)}, Q_2^{(2s)}, Q_3^{(2s)}\cdots$ is completely uniformly distributed in the $2s$-dimensional unit cube. For a given completely uniformly distributed sequence $\{\alpha_n\}$ the author constructs a sequence $\{\beta_n\}$ such that the two sequences are jointly completely uniformly distributed.

Finally the author constructs a sequence $\{\varepsilon_n^{(2)}\}$ jointly normal with a given sequence $\{\varepsilon_n^{(1)}\}$ without the restriction $g_1=g_2$. *R. A. Rankin* (Glasgow)

4947:

Friedman, Bernard; and Niven, Ivan. The average first recurrence time. Trans. Amer. Math. Soc. **92** (1959), 25–34.

Let $\alpha\in(0, 1)$, $\varepsilon>0$, and define

$$t(\alpha, \varepsilon) = \inf\{n: |n\alpha-[n\alpha+\tfrac{1}{2}]|\leq\varepsilon, n>0\}.$$

The authors estimate certain mean values $\mu_j(\varepsilon)=\int_0^1 t^j d\alpha$, and prove in particular that for $\varepsilon\to 0$

$$\mu_1(\varepsilon)=\frac{6\log 2}{\pi^2}\varepsilon^{-1}+O(\log\varepsilon),$$

$$\mu_2(\varepsilon)=\frac{2\log 2+1}{\pi^2}\varepsilon^{-2}+O(\varepsilon^{-1}\log\varepsilon).$$

Similarly, for a point $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ in the k-dimensional unit cube C_k, defining t as the smallest positive integer such that all inequalities $|n\alpha_j-[n\alpha_j+\frac{1}{2}]|\leq\varepsilon$ hold, the authors show that there exist two positive constants c_1 and c_2 such that $c_1\varepsilon^{-k}<\mu_1<c_2\varepsilon^{-k}$, where μ_1 is the integral of t over C_k. *D. A. Darling* (Ann Arbor, Mich.)

4948:

Stepanov, B. V. On the mean value of the k^{th} power of the number of classes for an imaginary quadratic field. Dokl. Akad. Nauk SSSR **124** (1959), 984–986. (Russian)

The author shows that

$$\sum_{m=1}^{N} h^k(-m)=\frac{2^{k+1}N^{(k+2)/2}}{\pi^{k(k+2)}}Q_k+O\{N\exp[-C(\log N)^{\frac{1}{2}-\epsilon}]\},$$

where $h(-m)$ is the number of primitive quadratic forms of discriminant $-m$ and

$$Q_k=\sum_{n=1}^{\infty}(-1)^{(n-1)/2}\varphi(n)\tau_k(n^2)/n^3.$$

Estimates are made for $S=\sum_{m=1}^{N} L(1, \chi_m)^k$ where

$$L(1, \chi_m)^k=\sum_{n=1}^{\infty}\left(\frac{-m}{n}\right)\frac{\tau_k(n)}{n}.$$

(Here $\tau_k(n)$ is the number of decompositions of n as the product of k factors.) Writing $S=S_N+R_N$, the author uses

$$S_N = \sum_{m=1}^{N} \sum_{\substack{n=1\\(n,2m-1)}}^{N^{\alpha}} \left(\frac{-m}{n}\right) \frac{\tau_k(n)}{n},$$

for α real, whence $S_N=NQ_k+O(N^{\alpha/2+\epsilon})$ as shown by I. M. Vinogradov [*Izbrannye trudy*, Izdat. Akad. Nauk SSSR, Moscow, 1952; MR **14**, 610]. The main effort centers about R_N, which the author shows is estimated by the error in the main result, above. The work relies heavily on theorems and notation explained in the work of Rényi [Izv. Akad. Nauk USSR Ser. Mat. **12** (1948), 57–78; MR **9**, 413; p. 66], and A. I. Vinogradov [Dokl. Akad. Nauk SSSR **109** (1956), 683–686; MR **19**, 16]. The result corrects an earlier estimate of Dimman [Ž. Leningrad. Fiz.-Mat. Obšč. **1** (1927), 313–322]. *H. Cohn* (Tucson, Ariz.)

FIELDS

See also 4967.

4949:

★**Mignosi, Giuseppe. Sulla enumerazione delle radici della più generale equazione algebrica in un corpo finito.** Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 99–108. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

Let C be a finite field of $N=p^m$ elements, p odd. Let $f(x)=\sum_{i=0}^{n} a_i x^i$ be a polynomial of degree $n<N$ with coefficients in C. The following results are obtained. (1) The number ν of distinct roots of f in C is the degree of the polynomial $g(x)=\text{G.C.D.}(x^N-x, f(x))$. (2) Let $\alpha_{(r)}$ represent all positive integer decompositions $N-1=\alpha_0+\alpha_1+\cdots+\alpha_n$ such that $\alpha_1+2\alpha_2+\cdots+n\alpha_n=r(N-1)$. Then

$$\nu = \sum_{r=1}^{n} \sum_{\alpha_{(r)}} \frac{(N-1)!}{\alpha_0!\alpha_1!\cdots\alpha_n!}\, a_0^{\alpha_0} a_1^{\alpha_1}\cdots a_n^{\alpha_n}.$$

(3) The equation $x^n=a$ has roots in C if and only if $a^{(N-1)/\nu}=1$, where $\nu=\text{G.C.D.}(n, N-1)$. When this is satisfied, the roots are those of $x^\nu=a^\lambda$ for any integer λ such that $(n/\nu)\lambda\equiv 1 \bmod (N-1)/\nu$.

Results (1) and (3) are evident from the appropriate chapters of the modern algebra texts by van der Waerden, Albert, et al. Result (2) requires several cautions which the author fails to state. The inner sum is in the arithmetic of the field C, and yields either 0 or 1; the outer sum is ordinary (rational) arithmetic. Special exceptions must be made when one or more of the coefficients a_i is zero.

S. W. Golomb (Pasadena, Calif.)

4950:

Knobloch, Hans-Wilhelm; und Röhrl, Helmut. Zum Begriff der analytischen Fortsetzung in algebraischen Funktionenkörpern einer Veränderlichen. Math. Ann. **136** (1958), 187–200.

Let K be a separable algebraic function field in one variable with constant field L, and let X be the totality of places of K. Let each completion K_x of K with respect to $x\in X$ be identified with a subfield of the power series field $L'({}_xt)=K_x'$, where L' is an algebraic closure of L and ${}_xt$ is a local uniformizing parameter at x. For each (x, y) in a certain subset A of $X\times X$ let M_x^y be a subfield of K_x' containing K, and i_y^x a monomorphism of M_x^y into K_y' which is the identity on K and on $M_x^y\cap L'$. The authors call such a system $\{A, M_x^y, i_y^x\}$ a continuation schema if it satisfies certain conditions which represent an abstract formulation of three properties of classical analytic continuation for the germs of meromorphic functions on the Riemann surface X of K when L is the complex number field: continuation from place to place on X induces isomorphisms of respective completions of K; continuation and differentiation commute; a power series everywhere uniquely continuable induces an element of K.

Given a continuation schema F, the notion of sheaf of abelian groups (or modules) is defined (F itself, e.g., becoming the sheaf of germs of meromorphic functions on X), as are collateral sheaf-theoretic notions. Certain 0-dimensional cohomology groups are shown isomorphic to the groups of differentials of the first and second kinds on K. It is proved that for any such sheaf G there exist Q and N, with Q injective, such that $0\to G\to Q\to N\to 0$ is exact.

If L has characteristic zero, and if F satisfies a further condition involving the differentials of the second and third kinds, the authors are able to define abelian integrals of differentials of the second kind, and their periods. Elementary cycles and the first homology group of X are defined, and it is proved that a differential of K all of whose periods along such cycles vanish is of the second kind. Finally, in characteristic zero, the existence of a continuation schema having all the required properties is proved. *F. D. Quigley* (New Orleans, La.)

4951:

★**Krull, Wolfgang. Zur Theorie der Bewertungen mit nichtarchimedisch geordneter Wertgruppe und der nichtarchimedisch geordneten Körper.** Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, pp. 45–77. Centre Belge de Recherches Mathématiques. Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris; 1957. 293 pp. 250 francs belges.

Ce mémoire, consacré aux structures d'ordre non archimédiennes définies à partir d'un corps (soit directement, soit à partir de valuations) se compose de deux parties bien distinctes.

(I) Soient K un corps commutatif et $B^1, \cdots, B^n$ un nombre fini d'anneaux de valuation de K. Un théorème de Ribenboim [Math. Z. **68** (1957), 1–18; MR **19**, 1035] permet de caractériser ainsi le groupe de divisibilité H de $A=B^1\cap\cdots\cap B^n$: L'ensemble des anneaux de valuation de K, ordonné par inclusion, étant semi-réticulé supérieurement, on considère l'ensemble (fini) des anneaux de valuation de K de la forme $\sup(B^{\nu_1}, \cdots, B^{\nu_s})$. Les valuations correspondantes permettent alors de définir un système projectif de groupes totalement ordonnés dont le groupe réticulé H est la limite projective.

De manière analogue, si G est un groupe abélien réticulé n'ayant qu'un nombre fini de t-idéaux premiers maximaux distincts, on peut plonger G dans une limite projective H de groupes totalement ordonnés. L'auteur montre un cas dans lequel G est somme directe ordonnée de groupes

totalement ordonnés et pose le problème suivant: A t'on toujours $G=H$? Le rapporteur peut répondre par l'affirmative.

(II) Soient K un corps totalement ordonné dans lequel tout élément >0 admet une racine carrée, B la valuation naturelle de K et $\bar{K}$ le corps résiduel. U étant un espace vectoriel de dimension finie sur K euclidien (en un sens évident) avec une norme notée $|x|$, on considère l'ensemble C [resp. C'] formé par tous les x de U tels que $B(|x|)\geqq 0$ [resp. >0]. $\bar{C}=C/C'$ est un espace vectoriel sur $\bar{K}$.

Soit maintenant G un sous-groupe additif de U ayant un nombre fini de générateurs. En multipliant les éléments de G par un élément convenable de K on peut supposer $G\subset C$. Le groupe G sera dit D-maximal si son rang est égal à la dimension du K-espace vectoriel qu'il engendre. Lorsque K est archimédien, dire que G est D-maximal revient à dire qu'il est discret. Dans cette étude l'auteur démontre le résultat suivant qui généralise ceci dans le cas où K n'est plus supposé archimédien.

G est dit groupe minimal si toute suite (f_n) d'éléments de G telle que $|f_n|\geqq|f_{n+1}|$ (pour chaque n) est telle qu'à partir d'un certain rang on ait $|f_n|=|f_{n+1}|$. A chaque G sont associés de manière unique une suite de groupes $G_0=(0)\subset G_1\subset\cdots\subset G_s=G$, et une suite $\alpha_1,\cdots,\alpha_s$ d'éléments de K tels que: (1) $g\in G_i$ entraîne $B(|g|)\geqq B(\alpha_i)$, et (2) si $g\in G_i$, $g\notin G_{i-1}$, $B(|g|)=B(\alpha_i)$ $(1\leqq i\leqq s)$. Soient $\mathfrak{G}_i$ l'espace engendré par G_i et F_i l'ensemble des éléments de $C\cap\mathfrak{G}_i$ orthogonaux à $\mathfrak{G}_{i-1}$. Soit $\overline{F_i'}$ le sous-espace de $\bar{C}$ engendré par $\overline{\alpha_i^{-1}G_i}$. Si on a $\overline{F_i}=\overline{F_i'}$ $(1\leqq i\leqq s)$ le groupe G sera dit normal. Il sera dit hypernormal s'il est normal et si, h^i désignant la projection de g sur $\mathfrak{G}_i$, on a $B(|h^i|)\geqq B(\alpha_i)$ $(1\leqq i\leqq s)$ pour tout $g\in G$. Ceci posé, on a le résultat: Pour que G soit minimal, il faut et il suffit qu'il soit D-maximal et hypernormal. *P. Jaffard* (Paris)

4952:

★Krasner, Marc. Approximation des corps valués complets de caractéristique $p\neq 0$ par ceux de caractéristique 0. Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, pp. 129–206. Centre Belge de Recherches Mathématiques. Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris; 1957. 293 pp. 250 francs belges.

The author begins this memoir which is an amplification of some of his notes in the Comptes Rendus dating back to 1947, with observations on "multiplicative congruences" in a field k possessing a rank one valuation. Thus, suppose that A is an ideal in k; then $\beta/\alpha\equiv 1 \pmod A$ is to mean $\beta/\alpha\equiv 1 \pmod A$ in the customary additive sense, i.e., $|\beta/\alpha-1|\leqq|A|$, where $|\cdots|$ denotes the absolute value of k, and $|A|$ is essentially $\sup_{a\in A}|a|$ with the provision that $|A|=\Pi$ (multiplicative divisor) is the real number r if A is principal and the "semi-real" number $(r,-)$ if A is not principal. It is easy to see that the multiplicative congruence classes mod Π form (I) a set $k/\Pi=\{C_\alpha, C_\beta,\cdots\}$ which is a commutative pseudogroup with respect to the obvious multiplication of classes and (II) a commutative hypergroup with respect to addition of classes (looking at them as subsets of k), i.e., the sum $C_\alpha+C_\beta$ equals the set of all classes contained in the set theoretical sum of C_α and C_β. Furthermore, an absolute value $|C_\alpha|$ is uniquely defined by $|\alpha|$, $\alpha\in C_\alpha$. Next the author defines by abstraction from the above the concept of hyperfield H with a valuation $|\cdots|$. He postulates that the set H admit a multiplication xy, x, y in H, a "sum" $x+y$, in general consisting of a subset of H, such that (I), (II) hold; also (III) H is to be distributive, (IV) $1+0=1$ for the identity elements, and for each x there shall be a unique y with $0\in x+y$. Furthermore, there is to be defined a real-valued absolute value $|x|\geqq 0$ on H satisfying (A) $|xy|=|x||y|$, (B) $|x+y|\leqq \operatorname{Max}(|x|,|y|)$ (note that $x+y$ is in general a set of real numbers), (C) if $0\notin x+y$, then $|x+y|$ is a single real number; (D) there is a semi-real number ρ such that for $d(z,z')\leqq\rho\operatorname{Max}(|x|,|y|)$ the relation $z\in x+y$ implies $z'\in x+y$ (for all x, y), the distance $d(z,z')=|z-z'|$ being a single real number if $z\neq z'$ (in the preliminary discussion such a ρ is $|A|=|\Pi|$; $\rho=N(H)$ is called the norm of the hyperfield; if $\rho=0$, then H is a field). As in the case of a field, multiplicative congruences (modulo a semi-real number Π) can be introduced in H. The above axioms are set up such that the collection of cosets modulo Π, H/Π, is again a hyperfield with a valuation. The sets H/Π are seen to be, within suitably defined isomorphisms, the only homomorphic images (valued hyperfields) of H. Therefore it is natural to investigate projective sequences (and their limits) of valued hyperfields $H_{i+1}\rightarrow H_i$ (obviously $N(H_i)\geqq N(H_{i+1})$) subject to $N(H_i)\rightarrow 0$. It follows in a straightforward manner that sequences (projective for the given homomorphisms) form a field with a valuation such that for each i there is a Π_i, $=$real ρ_i or semi-real $(\rho_i,-)$, in the value set of k with $H_i\cong k/\Pi_i$. Next, two fields with valuations, k and k', are termed residually isomorphic with norm $\rho>0$ if there exist multiplicative divisors Π and Π' of norm ρ in k and k', respectively, such that the corresponding hyperfields k/Π and k'/Π' are isomorphic. Then a sequence of fields k_i with valuations is called convergent if k_i and k_{i+1} are residually isomorphic with norm ρ_i so that $\rho_i\rightarrow 0$. By the preceding results the hyperfields k_i/Π_i are seen to form a projective sequence which has a limit field k. This field is called the limit of the sequence of fields. This concept generalizes the interpretation of the valuation ring of a complete field as the projective limit of residue class rings for ideals A_i with $\bigcap_i A_i=0$ $(k=k_i$ for all $i)$. A simple example shows that fields of non-zero characteristic may be viewed as limits of fields of zero characteristic. It is announced that every complete field of nonzero characteristic k can be considered as a limit of fields of zero characteristic k_i. A result of this type enables the author to reduce the theory of extensions of the field k to that of extensions of sufficiently close approximating fields k_i—a result which is not too surprising since, for example, the elements of the higher ramification groups are characterized by congruences, and thus are amenable to description in pseudofields. The author uses generalizations of Hensel's Lemma (conditions on the coefficients of defining polynomials for extensions so that the latter are isomorphic). His formulations are stated in terms of approximating hyperfields, their extensions, and essentially reflect properties of resultants and discriminants of polynomials in fields. *O. F. G. Schilling* (Chicago, Ill.)

ALGEBRAIC GEOMETRY

See also 4950.

4953:

Severi, Francesco. Nuove relazioni fra il genere aritmetico d'una ipersuperficie generale A tracciata sopra

una varietà algebrica e i generi aritmetici delle varietà caratteristiche di A. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **26** (1959), 3–5.

The author shows that if A is a topologically general hypersurface on a variety M_r of dimension r, and $|B|$ is the adjoint system of A, then

$$[1+(-1)^{r-1}]p_a{}^r+p_a{}^{r-1}-r-1 = \sum_{a=1}^{a=r} (-1)^{r-a}p_a{}^{r-a}(B^a),$$

where $p_a{}^r$, $p_a{}^{r-1}$, $p_a{}^{r-a}(B^a)$ are the arithmetic genera of M_r, A and B^a. *J. A. Todd* (Cambridge, England)

4954:

Marchionna Tibiletti, Cesarina. Un complemento al teorema d'esistenza di Riemann. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 240–249.

The existence theorem referred to in the title is that every Riemann surface, defined by assigned (consistent) branching permutations at assigned points in the x-plane does in fact represent the branching of some algebraic function $y(x)$ defined by a polynomial equation $f(x, y)=0$. The theorem proved here is that if n is the number of sheets and p the genus of the Riemann surface, for every value of $\nu \geqq \max(n, 2p-2)$, $f(x, y)$ can be chosen of order ν in (x, y), and such that the curve $f(x, y)=0$ has a $(\nu-n)$-ple point with $(\nu-n)$ distinct tangents at infinity on the y-axis, and no other singularities except double points with distinct tangents. *P. Du Val* (London)

4955:

Boseck, Helmut. Zur Theorie der Weierstrasspunkte. Math. Nachr. **19** (1958), 29–63.

Let R be a field of algebraic functions of one variable over the field K and let $\mathfrak{p}$ be a place of R/K. A positive integer φ is called a deficiency index for $\mathfrak{p}$ in case there is no function in R which (as a divisor) is an integral multiple of $\mathfrak{p}^{-\varphi}$. In the classical case (K is the field of complex numbers) if R is of genus g, the integers $1, 2, \cdots, g$ are the deficiency indices for all but a finite number of exceptional places $\mathfrak{p}$. These exceptional places are the Weierstrass points of R. In case K is of positive characteristic, F. K. Schmidt has observed [Math. Z. **45** (1939), 75–96] that there exist fields R/K with infinitely many Weierstrass points in this sense, and he proposed that when K is algebraically closed, a place $\mathfrak{p}$ of R/K be called a Weierstrass point or an ordinary point according as there exist finitely or infinitely many places of R/K with the same deficiency indices as $\mathfrak{p}$. This definition avoids the question as to whether or not there is a fixed set of integers $\{\varphi_1, \varphi_2, \cdots, \varphi_g\}$ the elements of which are the deficiency indices for almost all places $\mathfrak{p}$ of R/K. The present paper is addressed to this question in the case where K is a perfect field.

Let $\omega_1, \omega_2, \cdots, \omega_g$ be independent differentials of first kind, and let $y_i=\omega_i/\omega_1$, $i=1, 2, \cdots, g$, let z be a separating transcendence base for R/K and let D be the derivation with respect to z. There exist integers $\mu_1(=0)$, $\mu_2, \cdots, \mu_g$ such that $|D^{\mu_i}(y_j)|\neq 0$ and which are minimal in the lexicographic order with respect to this property [F. K. Schmidt, Math. Z. **45** (1939), 62–74]. It is shown that if $\varphi_i=\mu_i+1$, $i=1, 2, \cdots, g$, then all but a finite number of places $\mathfrak{p}$ of degree one of R/K have $\varphi_1, \varphi_2, \cdots, \varphi_g$ as deficiency indices, which are then termed the deficiency indices of R/K. Moreover, if L is an algebraic extension of K and $R(L)$ is the corresponding extension of R, then $R(L)/L$ has the same set of deficiency indices as does R/K. A place $\mathfrak{p}$ of degree d of R/K is then a Weierstrass point in case there is a ground field extension $R(L)$ of R in which a place $\bar{\mathfrak{p}}$ of first degree splits from $\mathfrak{p}$ such that the deficiency indices of $\bar{\mathfrak{p}}$ differ from those of R/K. The last section of the paper is devoted to a detailed study of the Weierstrass points of cyclic function fields over perfect fields K.
H. T. Muhly (Iowa City, Iowa)

4956:

Roquette, Peter. Über den Riemann-Rochschen Satz in Funktionenkörpern vom Transzendenzgrad 1. Math. Nachr. **19** (1958), 375–404.

The Rosenlicht generalization of the Riemann-Roch theorem for algebraic function fields of one variable [Ann. of Math. **56** (1952), 169–191; MR **14**, 80] is here given a conceptual formulation as an isomorphism theorem. This formulation leads to an analogue which is apparently new. Let K be a field of algebraic functions of transcendence degree one over the ground field k and let R be a ring between k and K which has K as quotient field. Let V_R be the k-module of all R-repartitions in the sense of Chevalley-Rosenlicht, let i_R be the natural injection of R into V_R and let π_R be the natural projection of V_R onto the k-module $R^*=V_R/i_R(R)$. If $\mathfrak{a}$ is an R-divisor let $V_R(\mathfrak{a})$ denote the k-module of elements of V_R that are divisible by $\mathfrak{a}$, let $R(\mathfrak{a})$ denote the inverse image $i_R{}^{-1}(V_R(\mathfrak{a}))$, and let $R^*(\mathfrak{a})$ denote the factor module $R^*/\pi_R(V_R(\mathfrak{a}))$. If both $\mathfrak{a}$ and $\mathfrak{b}$ are R-divisors and if $\mathfrak{a}$ is an integral multiple of $\mathfrak{b}$, it is possible to define in a natural way a sequence of mappings,

$$0 \to R(\mathfrak{a}) \to R(\mathfrak{b}) \to V_R(\mathfrak{b})/V_R(\mathfrak{a}) \to R^*(\mathfrak{a}) \to R^*(\mathfrak{b}) \to 0,$$

which can be proved to be exact. The statement of the existence of this exact sequence is called the "divisor theorem" by the author. If the k-modules $R(\mathfrak{a})$ and $R^*(\mathfrak{a})$ are known to be finite-dimensional, the usual numerical form of the Riemann-Roch theorem can be obtained by equating the alternating sum of the dimensions of the modules in the divisor theorem sequence to zero and noting that $\dim V_R(\mathfrak{b})/V_R(\mathfrak{a})=\text{degree}(\mathfrak{a})-\text{degree}(\mathfrak{b})$.

In the formulation of the divisor theorem the ring R is fixed and modules relative to two divisors are considered. However, one may also consider two rings R and S ($R\subset S$) and a fixed R-divisor $\mathfrak{a}$ that is of course also an S-divisor. The analogue of the divisor theorem mentioned above is the "ring theorem", which asserts the existence of an exact sequence of mappings,

$$0 \to R(\mathfrak{a}) \to S(\mathfrak{a}) \to (\bar{R} \cap S)/R \to R^*(\mathfrak{a}) \to S^*(\mathfrak{a}) \to 0,$$

in which $\bar{R}$ denotes the integral closure of R in K. The proof of the ring theorem is combined with the proof of the "finiteness theorem" which asserts that $R(\mathfrak{a})$, $R^*(\mathfrak{a})$, and $\bar{R}/R$ are finite-dimensional k-modules. If $\delta(R)=\dim \bar{R}/R$, and if $\chi_R(\mathfrak{a})$ is the Euler characteristic of $\mathfrak{a}$ relative to R, then by equating the alternating sum of the dimensions of the terms of the ring theorem sequence to zero, the relation $\chi_R(\mathfrak{a})-\chi_S(\mathfrak{a})=\delta(S)-\delta(R)$ is obtained. The paper closes with a series of applications and a sketch of extensions of the divisor and finiteness theorems to the case in which the role of R is taken over by a finite torsion-free R-module. *H. T. Muhly* (Iowa City, Iowa)

4957:

Cartier, Pierre. Questions de rationalité des diviseurs en géométrie algébrique. Bull. Soc. Math. France **86** (1958), 177–251.

This is the first detailed report on the author's contribution to algebraic geometry, and provides us with a new method of dealing with the peculiarity of the modular case. Applications to abelian varieties are postponed to later publication.

Chapter 1: A k-variety (=an abstract variety defined over k, in Weil's terminology) is defined intrinsically. (In order to get a manageable definition, it seems necessary to modify his axiom a little.) Basic notions and theorems in algebraic geometry, e.g., on rational mappings and scalar extensions, are stated without proofs.

Chapter 2: The following (proposition 3) may be regarded as the main theorem in this chapter. Let K be a field of characteristic p, and let $\mathfrak{g}$ be a p-Lie-algebra of derivations of K with $\dim_K \mathfrak{g} < \infty$. Let V be a vector space over K. Assume that a mapping $r: \partial \rightarrow r(\partial)$ from $\mathfrak{g}$ into the endomorphism ring of the additive group V is given, satisfying $r(a\partial)(v) = a \cdot r(\partial)(v)$, $r(\partial)(av) = \partial(a)v + a \cdot r(\partial)(v)$ ($v \in V$, $a \in K$), and the conditions for homomorphism of a p-Lie-algebra. Then $V = K \otimes_{K_\mathfrak{g}} V_\mathfrak{g}$, where $V_\mathfrak{g} = \{v \mid v \in V, r(\partial)(v) = 0 \text{ for all } \partial \in \mathfrak{g}\}$ and $K_\mathfrak{g}$ is the field of $\mathfrak{g}$-constants. In particular, $V_\mathfrak{g} \neq 0$.

The proof is quite elementary, and the use of the theory of simple rings is avoided. At the same time, the Galois theory of purely inseparable extensions of height 1, due to Jacobson, is obtained. Then the well-known "Cartier operator" C on the closed differential forms of characteristic p [see C. R. Acad. Sci. Paris **244** (1957), 426–428; MR **18**, 870] is introduced. The main property of C is that, for $\omega \in \Omega^1$, we have

$$\omega = dx \Leftrightarrow d\omega = 0, \quad C\omega = 0;$$
$$\omega = dx/x \Leftrightarrow d\omega = 0, \quad C\omega = \omega.$$

Chapter 3: Serre's theory of coherent sheaves is generalized to the case where the ground field is not algebraically closed, and the effect of ground field extensions is discussed.

Chapter 4: Let X be a k-variety, let R_k^* be the multiplicative group of non-zero rational functions on X, let $\mathcal{R}_k^*$ be the constant sheaf with fibre R_k^*, and let $\mathcal{O}^*$ be the sheaf of the unit groups of the local rings on X. Then a k-divisor D is defined as a global section of the sheaf $\mathcal{R}_k^*/\mathcal{O}^*$. That is, a divisor is a divisorial cycle which has a local equation defined over k at every point of X. If k' is a finite extension of k such that $k'^p \subset k$, then $\mathfrak{g} = \mathfrak{g}(k'/k)$ can be identified with $\mathfrak{g}(R_{k'}/R_k)$, where $R_{k'}$ and R_k are rational function fields of X over k' and k, respectively. Put $\partial^*(f) = \partial f/f$ ($f \in R_{k'}^*$). Then ∂^* induces a homomorphism ∂_1^* from the group of the k'-divisors into $H^0(X, \mathcal{R}_{k'}/\mathcal{O}_k)$, $\mathcal{R}_{k'}$ and $\mathcal{O}_k$ being the sheaves with fibre $R_{k'}$ and local rings over k, respectively. The main result is: A necessary and sufficient condition for a k'-divisor D' to be k-rational is that $\partial_1^*(D') = 0$ for all $\partial \in \mathfrak{g}$.

A similar criterion for divisor classes is also given.

M. Nagata (Kyoto)

4958:

Taniyama, Yutaka. Distribution of positive 0-cycles in absolute classes of an algebraic variety with finite constant field. Sci. Papers Coll. Gen. Ed. Univ. Tokyo **8** (1958), 123–137.

Let V be a projective variety, normal over a field k and non-singular in codimension 1. Let α be a mapping of V into an abelian variety A, defined over k and such that it cannot be factored non-trivially as $\alpha = \lambda \circ \beta$, where β is a mapping of V into an abelian variety B and λ is an isogeny of B onto A. Let r and a be the dimensions of V and A, respectively. Let $V^{(m)}$ be the m-fold symmetric product of V; every positive 0-cycle $\mathfrak{a} = P_1 + \cdots + P_m$ of degree m on V determines a point of $V^{(m)}$, and, by putting $\alpha(\mathfrak{a}) = \sum_i \alpha(P_i)$, one defines a mapping α_m of $V^{(m)}$ into A. One main purpose of the present paper is to prove the following theorem: if $m \geqq 2a+1$, then, for all points y of A, the cycle $\alpha_m^{-1}(y)$ is defined, has only one (irreducible) component, and its degree (as a cycle in the ambient projective space) is independent of y; moreover, if A is the Albanese variety of V, α the canonical mapping of V into A, and y a generic point of A, then the unique component of $\alpha_m^{-1}(y)$ is of multiplicity 1.

The method of proof, as novel as it is surprising, consists in first considering the case when k is the finite field with q elements; if k_ν is the extension of k of degree ν, and if the number of rational points over k_ν on a closed algebraic set, known to have no component of dimension $<d$, is asymptotically equal to $q^{d\nu}$, then that set can consist of only one component of dimension d. In applying this idea to the situation described above, essential use is made of the isogeny $x \rightarrow x^{q^\nu} - x$ of A onto itself, whose kernel g_ν consists of the points of A which are rational over k_ν; this may be viewed as defining an abelian non-ramified covering of A, isomorphic to A, whose Galois group is isomorphic to g_ν, and therefore, by "pull-back" by means of α, a similar covering of V, hence also of "almost all" the curves obtained by intersecting V with linear varieties of the appropriate dimension. Once the theorem is obtained for finite groundfields, the method of reduction modulo $\mathfrak{p}$ enables the author to extend it to the general case. At the same time, his estimates for finite ground fields yield valuable new information concerning zeta-functions, in the direction of the conjecture according to which the number of rational points of V over k_ν is uniquely determined, up to a remainder term of order $O(q^{(r-1)\nu})$, by r and by the zeta-function of the Albanese variety of V. A final corollary says that, for any groundfield k, the Albanese variety of V remains such under reduction modulo almost all valuations of k.

A. Weil (Princeton, N.J.)

4959:

★**Lang, Serge. Abelian varieties.** Interscience Tracts in Pure and Applied Mathematics. No. 7. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London; 1959. xii+256 pp. $7.25.

This book deals with the theory of general Abelian varieties and also that of Albanese and Picard varieties of given varieties. It is chiefly based on the lectures given by A. Weil during 1954–1955 (together with the author's own contribution). Also Chow's work on the trace and image is treated in the last chapter.

Generally speaking, this book is very well written and would give investigators an excellent account of what has been done, without going through many papers. Also it is convenient that this book contains all results in Weil's

book on Abelian varieties [*Variétés abéliennes et courbes algébriques*, Actualités Sci. Ind. no. 1064, Hermann, Paris, 1948; MR **10**, 621] (except possibly a construction of a group variety from a variety having a normal law of composition); some proofs are simplified and are made lucid. However, it is regrettable that some of the basic and useful theorems on Abelian varieties (such as duality, absence of torsion, Riemann-Roch theorem, theorem of Frobenius, etc.) had to be omitted (partly because these were not available at the time when the book was being prepared). The following list of chapter headings and comments will make the scope of this book clear.

In the following, we assume that all ambient varieties are non-singular in co-dimension 1. Chapter I contains preparatory remarks about group varieties, and some of them have no proofs (references are given). It might have been a good idea, in order to make this book self-contained, to include at least a construction of a group variety from a variety with a normal law of composition. Contents of Chapter II are: properties of rational mappings of varieties into Abelian varieties (which include the Poincaré complete reducibility theorem); construction of Jacobian varieties of curves; construction of Albanese varieties of given varieties. Chapter III starts with the definition of algebraic equivalence of cycles and gives a proof of Weil's theorem of the square: $X(a, b)-X(a, b')-X(a', b)+X(a', b')\sim 0$ on a product $U\times V\times W$. From this, it follows that if X is a divisor on $G\times W$, where G is a group variety, $X(a_1)-X(a_2)-X(a_3)+X(a_4)\sim 0$ if $a_1a_3^{-1}=a_2a_4^{-1}$, which is a fundamental theorem in the theory of Picard varieties as treated here. After this, the symbol φ_X is defined for a divisor X on a commutative group variety G, as a homomorphism $a\to \mathrm{Cl}(X_a-X)$ of G into the group of divisor classes Pic (G) of G with respect to linear equivalence. Then the theorem of the square shows that the kernel of φ_X is the algebraic subgroup of G. Chapter IV begins by proving an existence of a positive non-degenerate divisor X on an Abelian variety A^r (X is such that the kernel of φ_X is finite). From this, the Picard variety of A is constructed. This chapter contains a proof of the theorem that $\nu(\sum m_i\alpha_i)$ is homogeneous of degree $2r$ in the m_i with rational coefficients (in particular $\nu(n\delta)=n^{2r}$), together with related topics, which is simpler than the one given by Weil originally and is based on the author's own contribution. In Chapter V, first the transpose ${}^t\alpha$ of a homomorphism α of an Abelian variety A into another Abelian variety B is defined, then some related formulas are proved. Next, assuming $A=B$, an involution $\alpha\to\varphi_X^{-1}\,{}^t\alpha\,\varphi_X=\alpha'$ is given and the Castelnuovo inequality $\mathrm{tr}(\alpha\alpha')>0$ (if $\alpha\neq 0$) is proved. From this, it is proved that if A is defined over a finite field k with q elements, absolute values of characteristic roots of the Frobenius endomorphism relative to k are all equal to $\sqrt{q}$. The chapter ends with a few remarks about positive endomorphisms. Chapter VI begins with the existence theorem of the Picard variety of a given variety. Then the theorem of divisorial correspondences is given. Using this and results of Chapter V, a proof of the Riemann hypothesis for curves over finite fields is given. Finally, the reciprocity theorem $f((g))=g((f))$ on curves is generalized to Abelian varieties. In Chapter VII, l-adic representations of homomorphisms are discussed, more neatly than in Weil's book. As a consequence, the structure of the module of homomorphisms is given, among other things. The chapter ends with a remark about a polarized Abelian variety. In Chapter VIII, Chow's theory of K/k-image and K/k-trace of an Abelian variety defined over K is given. It also contains results about exact sequences, which are sometimes useful. Finally, an Appendix is added to include some auxiliary results about correspondences used in this book. Readers will find an "Historical Note" at the end of each chapter, about works of algebraic geometers, mainly those which were done after Weil's *Foundations of algebraic geometry* [Amer. Math. Soc., New York, 1946; MR **9**, 303] had appeared. This might be helpful sometimes. Also some problems are mentioned. {Remarks: In the Note to Chapter VI, the theory of correspondences, as formulated here, is attributed to Severi, but at least Hurwitz's name should have been mentioned also, since the theory of correspondences itself is essentially due to him (expressed in terms of Abelian integrals). Also, elsewhere, a remark is made that the seesaw principle is implicit in Severi's work, but as far as the reviewer knows, it was rather explicit.}

T. Matsusaka (Evanston, Ill.)

4960:

Lang, Serge; and Tate, John. Principal homogeneous spaces over abelian varieties. Amer. J. Math. **80** (1958), 659–684.

Let K/k be a Galois extension with the Galois group $G(K/k)$ (possibly infinite), A be an algebraic group defined over k, $A(K)$ be the group of points of A which are rational over K. A cochain $a(\alpha_1, \cdots, \alpha_r)$ of $G(K/k)$ with values in $A(K)$ is of finite type if there is a finite subextension F such that it depends only on the effects of the automorphisms α_i on F. Denoting by $C(K/k, A)$ the complex formed by these finite cochains, let $H^r(K/k, A)$ be its cohomology groups (sets, if A is not commutative). First, most results in Galois cohomology theory are extended to this case (for more general functor $A(K)$). Then using Weil's result on the field of definition of an algebraic variety, it is shown that there is a canonical bijection of $H^1(K/k, A)$ to the set of classes of k-isomorphic principal homogeneous spaces for A which have rational points over K. This generalizes Chatelet's result.

Further study of $H^1(K/k, A)$ is made, assuming that A is an Abelian variety defined over k. First, assuming that k is a local field, $H^1(K/k, A)$ is studied in relation with the reduction mod p. Passing to m-global fields (number fields, function fields over algebraically closed ground fields, fields of finite type, etc.; a characterization of m-global fields is given) ($m\not\equiv 0 \bmod p$, p being the characteristic of k), and showing that $A(k)/mA(k)$ is finite, the finiteness of the m-primary part of $H^r(K/k, A)$ is proved when r is positive and K/k is finite. This is used to show that the group of elements of $H^1(K/k, A)$ which split at all primes has the property that its subgroup of elements of order m ($m\not\equiv 0 \bmod p$) is finite. Finally, a theorem is given which shows that when k is a global field $H^1(K/k, A)$ is a large group.

T. Matsusaka (Evanston, Ill.)

4961:

Šafarevič, I. R. The group of principal homogeneous algebraic manifolds. Dokl. Akad. Nauk SSSR **124** (1959), 42–43. (Russian)

The author puts together without proofs some theorems about the group $H(\alpha, k)$ of the principal homogeneous

algebraic manifolds, connected with an abelian manifold α, introduced by A. Weil [Amer. J. Math. **77** (1955), 355–391, 493–512; MR **17**, 533]. *W. Burau* (Hamburg)

LINEAR ALGEBRA

See also 5415, 5515.

4962:

Lehti, Raimo. Eine Methode von sukzessiven Projektionen zur Lösung der linearen algebraischen Vektorgleichung und ihre Anwendungen für Inversion von Matrizen. Soc. Sci. Fenn. Comment. Phys.-Math. **21** (1958), no. 5, 35 pp.

On a geometrical basis, also described in the paper reviewed below, the author develops a method of solving a regular system of linear equations, making use of the fact that a solution of such a system, and in reality even the inverse of the matrix of the coefficients, is known if one has a reciprocal basis to the system of the column vectors of this matrix. The given basis of left-vectors $\langle a_j|$ represents n linearly independent points, thus an n-vertex polyhedron, called "tetrahedron" by the author. The reciprocal basis $|a^i\rangle$ of right-vectors then represents the same "tetrahedron", defined by its hyperplane sides. The problem consists in finding the vertices of the figure if the side planes are given. This is solved by a succession of projections, and the geometrical process is then carried out algebraically. In order to make the method applicable for numerical work the left-vectors are represented by row vectors, the right-vectors by columns. The method is illustrated by a number of numerical examples. Finally there is a counting of elementary operations required for this process of inversion, similar to that proposed by Bodewig. {A reader not absolutely fluent in German may have occasional difficulties in following the author's arguments.} *H. Schwerdtfeger* (Montreal, P.Q.)

4963:

Lehti, Raimo. Tensorrechnung und projektive Geometrie. I. Grundlagen der Multivektoralgebra und Inzidenzgeometrie. Soc. Sci. Fenn. Comment. Phys.-Math. **21** (1958), no. 6, 86 pp.

This memoir, mainly expository, is planned as another elaboration of the proposition that linear algebra and projective geometry have the same structure [in the sense of R. Baer, *Linear algebra and projective geometry*, Academic Press, New York, 1952; MR **14**, 675]. It begins with a definition of a finite-dimensional vector space $\langle R|$ over a field K, and its dual, denoted by $|R\rangle$. The elements $\langle a|$ of $\langle R|$ are called left-vectors, those $|b\rangle$ of $|R\rangle$ right-vectors. Further there is the incidence product $\langle a|b\rangle$ for which the usual properties of the inner product are postulated. Two bases, viz. $\langle e_1|, \cdots, \langle e_n|$ of $\langle R|$ and $|e^1\rangle, \cdots, |e^n\rangle$ in $|R\rangle$, are said to be reciprocal if $\langle e_j|e^i\rangle=\delta_j^i$. The spaces $\langle R|$ and $|R\rangle$ are turned into projective spaces over K by the assumption that the vectors $a\neq 0$ (i.e., $\langle a|$ or $|a\rangle$, respectively) and λa ($\lambda\in K$, $\lambda\neq 0$) represent the same point. With regard to the incidence relation $\langle x|b\rangle=0$ the right vector $|b\rangle$ is said to be representative of a hyperplane in $\langle R|$. Thus, apart from notations, one has the usual basis for an analytic projective geometry. The second chapter studies "vector products and multivectors" in terms of "multilinear mappings" $y=f(x_1, \cdots, x_k)$ of the n-dimensional vector space V_x^n into a vector space V_y^m, where $x_i\in V_y^n$, $y\in V_j^m$. The mappings to be considered here are generalizations of the notion of determinant where the image space V_y is one-dimensional. The mapping f is said to be definite in the case that $f(x_1, \cdots, x_k)=0$ if and only if $x_1, \cdots, x_k$ are linearly dependent; and f is alternating if $f(\cdots, x_i, \cdots, x_j, \cdots)=-f(\cdots, x_j, \cdots, x_i, \cdots)$, invertible if from $f(x_1', \cdots, x_k')=f(x_1, \cdots, x_k)\neq 0$ it follows that $x_i'=\sum_{j=1}^k \lambda_i^j x_j$ and $\det(\lambda_i^j)=1$. It is shown that every invertible, non-vanishing mapping is definite. f is said to be maximal if the image space V_y of V_x^n under f ($V_y\subset V_y^m$) has maximum dimension $\dim V_y=\binom{n}{k}$. Any maximal mapping is invertible. If $y=f(x_1, \cdots, x_k)$ and $z=g(x_1, \cdots, x_k)$ are two multilinear mappings, f maximal and g alternating, then $z=Ly$ and L is a linear mapping. With respect to this principal theorem the following definition is justified: For a maximal mapping f the vector $x^k=f(x_1, \cdots, x_k)$ is said to be the outer product of the vectors $x_1, \cdots, x_k$ and is denoted by $x^k=x_1\wedge x_2\wedge\cdots\wedge x_k$. The x^k are "multi-vectors of degree k" or "k-vectors". The outer product is multilinear (distributive), associative, and anti-commutative: $x^s\wedge x^t=(-1)^{st}x^t\wedge x^s$. The bivector space has the dimension $\binom{n}{2}$, the n-vector space is one-dimensional. The vector space V_x^n itself is the multi-vector space of degree one, and the scalar space, of degree zero, has the dimension $\binom{n}{0}=1$; its vectors are the elements of the field K. For further developments and the geometrical interpretation the author refers to Grassmann, Forder, Bourbaki, Chevalley, and the recent book by Reichardt, *Vorlesungen über Vektor- und Tensorrechnung* [VEB Deutscher Verlag, Berlin, 1957; MR **19**, 764]. {The equally relevant book by W. Graeub, *Lineare Algebra* [Springer, Berlin-Göttingen-Heidelberg, 1958; MR **20** #3883] is not mentioned.} The third chapter develops the algebra of the combined incidence- and outer vector products. In essence it reproduces a part of Grassmann's Ausdehnungslehre with product relations taking the place of Grassmann's determinant formulae. The last chapter, by introducing coordinates in the usual way, makes the bridge to Grassmann's original theory. Extensive use is made of reciprocal bases. The complements of linear manifolds and the "regressive product" [Forder, *Calculus of extensions*, University Press, Cambridge, England, 1941; MR **3**, 12; pp. 228–241] are introduced; using the incidence product for vector products it appears in the form of the symbol $(a|^k(\langle b|^t|e\rangle^n)$ where $|e\rangle^n=|e^1\wedge\cdots\wedge e^n\rangle$. In the final section the theory is specialized for the case of 3-dimensional projective geometry.
H. Schwerdtfeger (Montreal, P.Q.)

4964:

★Fazekas, F. Untersuchung mit Matrizen einiger Fragen azentrischer Gebilde zweiter Ordnung. Einige Arbeiten des Lehrstuhles für Mathematik im Lehrjahre 1956/7, pp. 47–53. Wissenschaftliche Veröffentlichungen der Technischen Universität für Bau- und Verkehrswesen in Budapest, Budapest, 1958. 80 pp.

The matrix algebra of surfaces and curves of the

second order is considered. In matrix form the general equation of a surface of the second degree is

$$\mathbf{r}^*\mathbf{A}\mathbf{r}+2\mathbf{a}^*\mathbf{r}+c = 0.$$

Conditions for surfaces with central points, central lines, central planes, axes of symmetry, and planes of symmetry are obtained. A similar study of second degree curves is also included. *D. E. Spencer* (Storrs, Conn.)

4965:

Olkin, Ingram. Inequalities for the norms of compound matrices. Arch. Math. **10** (1959), 241–242.

Let A denote any complex $n\times n$ matrix, adj A its adjoint, and $\|A\|$ its euclidean norm. H. Richter [Arch. Math. **5** (1954), 447–448; MR **16**, 106] showed that

$$\|\operatorname{adj} A\| \leqq n^{-(n-2)/2}\|A\|^{n-1};$$

and an alternative proof, depending on polar decomposition and an inequality for symmetric functions, was given subsequently by the reviewer [ibid. **7** (1956), 276–277; MR **18**, 460]. In the present paper the author uses a very similar method to demonstrate that, if $A^{(k)}$ denotes the kth compound of A, then the expression

$$\left[\|A^{(k)}\|^2\Big/\binom{h}{k}\right]^{1/k}$$

decreases as k increases. This contains, of course, Richter's inequality as a special case. *L. Mirsky* (Sheffield)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 4930, 4986, 5148.

4966:

Satô, Hazimu. Different Noetherian rings in some axiomatic relations. J. Sci. Hiroshima Univ. Ser. A **22** (1958/59), 1–14.

Let A and B be commutative noetherian rings with A a subring of B having the same unit as B. The author assumes that for arbitrary ideals $\mathfrak{a}$ and $\mathfrak{b}$ in A the following axioms are satisfied: (P_1) $\mathfrak{a}B\cap A=\mathfrak{a}$; (P_2) $(\mathfrak{a}:\mathfrak{b})B=\mathfrak{a}B:\mathfrak{b}B$; (P_3) $(\mathfrak{a}\cap\mathfrak{b})B=\mathfrak{a}B\cap\mathfrak{b}B$. Under these conditions, various connections between the prime divisors of an ideal $\mathfrak{a}$ in A and those of $\mathfrak{a}B$ are established. The author then considers the more general situation where A, A' and B are commutative noetherian rings such that A and A' are subrings of B each with the same unit as B. If $\mathfrak{a}$ and $\mathfrak{a}'$ are ideals in A and A', we assume that $(\mathrm{P}_1{}^*)$ $(\mathfrak{a}B+\mathfrak{a}'B)\cap A=\mathfrak{a}$ and $(\mathfrak{a}B+\mathfrak{a}'B)\cap A'=\mathfrak{a}'$ and $(P_2{}^*)$ each pair $(A/\mathfrak{a},\ B/(\mathfrak{a}B+\mathfrak{a}'B))$ and $(A'/\mathfrak{a}',\ B/(\mathfrak{a}B+\mathfrak{a}'B))$ satisfies (P_1), (P_2) and (P_3) mentioned above. In this situation various connections are established between the prime divisors of $\mathfrak{a}$ in A, $\mathfrak{a}'$ in A' and $\mathfrak{a}B+\mathfrak{a}'B$ in B.

The author then applies these results to the theory of multiplicity in local rings. Let S and S' be local rings which contain a common field k and such that the complete tensor product T of S with S' over k is noetherian. It is shown that the triple S, S' and T satisfy $(\mathrm{P}_1{}^*)$ and $(\mathrm{P}_2{}^*)$. Assume further that $\mathfrak{m}$ and $\mathfrak{m}'$ are the maximal ideals of S and S' respectively and $S/\mathfrak{m}$ is an algebraic extension of k. Then T is a complete semi-local ring. If we denote by $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$ the maximal ideals of T, we have that the $\mathfrak{M}_i$-primary components of $\mathfrak{m}T+\mathfrak{m}'T$ all have the same length c, and rank $\mathfrak{M}_i=\dim S+\dim S'$ for all i. Further, if $\mathfrak{a}$ and $\mathfrak{a}'$ are $\mathfrak{m}$- and $\mathfrak{m}'$-primary ideals of S and S', respectively, and if $e(\mathfrak{a})$, $e(\mathfrak{a}')$ and $e(\mathfrak{a}T+\mathfrak{a}'T)$ denote the multiplicities of $\mathfrak{a}$, $\mathfrak{a}'$ and $\mathfrak{a}T+\mathfrak{a}'T$ in S, S' and T, respectively, then $e(\mathfrak{a}T+\mathfrak{a}'T)=nce(\mathfrak{a})e(\mathfrak{a}')$.

Many of the proofs can be simplified by observing that if A is a noetherian subring of B (B not necessarily noetherian), then (P_2) and (P_3) follow from the assumption that B is A-flat. Further, if B is A-flat, then B/A is A-flat if and only if (P_1) holds.

M. Auslander (Waltham, Mass.)

4967:

Samuel, Pierre. Sur les anneaux gradués. An. Acad. Brasil. Ci. **30** (1958), 447–450.

This note is devoted primarily to establishing the following generalization of the usual theorem concerning characteristic polynomials of graded modules. Let $A=A_0+\cdots+A_n+\cdots$ be a graded noetherian ring such that A_0 is an artin ring (ring with minimum condition) and there are homogeneous elements $x_1, \cdots, x_q$ such that $A=A_0[x_1, \cdots, x_q]$. Also, let $E=E_0+\cdots+E_n+\cdots$ be a finitely generated graded A-module. Then the length $\chi(E_n)$ is finite for all n, and if d denotes the least common multiple of the degrees of the x_i, there exist d polynomials $F_k(n)$ $(k=0, \cdots, d-1)$ with rational coefficients such that $\chi(E_{dn+k})=F_k(n)$ for all large n. Further, the degree of each F_k is $\leqq q-1$.

The following application of this result is then given. Let A be an integral domain, v a discrete, rank 1 valuation of the field of quotients of A such that (a) the valuation ring of v contains A, and (b) the center of v is a maximal ideal $\mathfrak{m}$ of A. For all integers $n\geqq 0$, let $\mathfrak{a}_n$ denote the set of x in A such that $v(x)\geqq n$. After observing that the $\mathfrak{a}_i$ are a filtration for A, it is shown that if the residue class field of v is a finite-dimensional extension of $A/\mathfrak{m}$, then the graded ring $G(A)=A/\mathfrak{a}_1+\cdots+\mathfrak{a}_n/\mathfrak{a}_{n+1}+\cdots$ associated with the filtration is noetherian

M. Auslander (Waltham, Mass.)

HOMOLOGICAL ALGEBRA

4968:

Borevič, Z. I.; and Faddeev, D. K. Theory of homology in groups. II. Projective resolutions of finite groups. Vestnik Leningrad. Univ. **14** (1959), no. 7, 72–87. (Russian. English summary)

[For part I, see same Vestnik **11** (1956), no. 7, 3–39; MR **18**, 188.] Let G be a finite group and K a commutative unitary ring. Convention: All $K[G]$ modules to be considered, and in particular the K-modules, are supposed to be free and of finite rank over K. The $K[G]$ modules A and B are said to be equivalent if there exist $K[G]$-projectives P, Q such that $A+P$ and $B+Q$ are isomorphic. Suppose furthermore that K has the property that if a free K-module A admits a direct decomposition $B+C$ with B free, then C is also free. With these assumptions and terminology it turns out that if $0\leftarrow K\xleftarrow{\partial}\phi_0\xleftarrow{\partial}\phi_1\xleftarrow{\partial}\cdots$ is any projective $K(G)$-resolution of $K(G$ acting trivially on $K)$, the modules $\partial\phi_n$ are unique up to equivalence. As

a consequence, if G is a p-group and K a complete local ring with residue field of characteristic p, there is a resolution of K in which the $\partial\phi_n = \Omega_n$ are indecomposable and they are unique up to isomorphism. In case K is a galois field this permits expression of the cohomology groups of G in terms of the Ω_n. *W. T. van Est* (Leiden)

GROUPS AND GENERALIZATIONS

See also 4951, 5004, 5020, 5160, 5171, 5172, 5441, 5442.

4969:

Piccard, Sophie. Structure des groupes libres. Ann. Sci. École Norm. Sup. (3) **76** (1959), 1–58.

The homomorphism from a free group onto the group obtained by making it abelian and setting all nth powers, for some n, equal to one is studied in detail.

4970:

Yacoub, K. R. On general products of two finite cyclic groups one being of order 4. Proc. Math. Phys. Soc. Egypt. no. 21 (1957), 119–126. (Arabic summary)

Semi-special permutations are applied to the classification of all general products G of a cyclic group A with generator a of order m and of a cyclic group B with generator b of order 4; that is, $G=AB$ where $A \cap B=1$. See the author's paper, Proc. Glasgow Math. Assoc. **2** (1955), 116–123 [MR **17**, 11] for definitions. One category is determined by an integral parameter r where $ab=ba^r$ with $r^4\equiv 1 \bmod m$. If s is a second value of the parameter with $r\not\equiv s \bmod m$, then the two resulting products are isomorphic if and only if $s\equiv r^3 \bmod m$. A second category is determined by two integral parameters s and t. Then, m is even, $2s^2\equiv 2$, $4t(1+s)\equiv 0$, $2(1+t)(s-1)\equiv 0 \bmod m$, and the relations $ab=b^3a^{2t+1}$, $a^2b=ba^{2s}$ hold. Each such set of parameters leads to a general product of A and B, while each general product is associated with a parameter r or with a pair of parameters s, t. *F. Haimo* (St. Louis, Mo.)

4971:

Curzio, Mario. Sui sottogruppi di composizione dei gruppi finiti. Ricerche Mat. **7** (1958), 265–280.

Let G be a finite soluble group and $\varphi(G)$ the lattice of subnormal (accessible) subgroups of G. If A is a neutral element of $\varphi(G)$, then A is characteristic in G; moreover, if, for another subnormal subgroup B, $A\cap B$ is normal in $A\cup B$, then the indices of the former in A and in B are relatively prime. Suppose now that G is also a t-group, i.e., a group in which the property of being normal is transitive. Then A is neutral in the lattice of normal subgroups of G if and only if the indices of $A\cap B$ in A and B are relatively prime, where B is any other normal subgroup. This condition on A is always sufficient, provided G is a finite group. The lattice $\varphi(G)$ for an arbitrary finite group G is complemented if and only if G is the direct product of simple groups. If G is soluble, then $\varphi(G)$ is reducible if and only if G is a direct product of groups with relatively prime orders. *K. A. Hirsch* (London)

4972:

Bachman, George. Geometry in certain finite groups. Math. Z. **70** (1958/59), 466–479.

Let Σ be a set $\{H_\alpha\}$ of subsets of a group G. The elements of Σ are "points" of a geometry based on the following notions. A "motion" is a transformation $H_\alpha \to gH_\alpha$, $g\in G$. Let $\Gamma\subset\Sigma$; define $F(\Gamma)\equiv$ subgroup of G leaving Γ elementwise invariant; $F^*(\Gamma)\equiv$ subgroup of G leaving Γ invariant; $\Pi(H)\equiv$ subset of Σ left elementwise fixed by the entire set $H\subset G$. Then $\Pi(gHg^{-1})=g\Pi(H)$; $N_G(F(\Gamma))=F^*(\Gamma)$, where N_G means "normalizer in G". The closure Γ' of Γ is $\Pi F(\Gamma)$. The author sets up axioms: (i) $F(\Sigma)=$ identity of G; (ii) a one-point subset is closed; (iii) if Γ is closed, $F^*(\Gamma)$ acts transitively on Γ; and (iv) if Γ_1, Γ_2 are closed and have the same dimension, then $g\Gamma_1=\Gamma_2$ for some $g\in G$. Here "dimension Γ" is the smallest t such that $\{\alpha_1, \alpha_2, \cdots, \alpha_{t+1}\}'=\Gamma$ for some $\alpha_1, \cdots, \alpha_{t+1}$. A translation is a transformation of Σ with no fixed points.

Typical results are the following. Theorem 3.1: Let $\Sigma=(H, K)'$. (Σ is one-dimensional, $H\neq K$.) Then the conjugates of H in G are the points of Σ, and no two conjugates have (in G) a common element $\neq e$. Also $N_G(H)=H$. A theorem of Frobenius shows that G has a normal subgroup N which consists of translations of Σ; $G=HN$. This subgroup consists of e plus all elements not in H or any of its conjugates. (H, N) form a "Frobenius pair". Theorem 3.2: Let $G=H_1N_1=H_2N_2$, where (H_i, N_i) form a "F.p." Then H_1 is contained in a conjugate of H_2, or vice versa. Examples of one-dimensional groups: dihedral and generalized dihedral groups of orders $2p$, dp $(d\,|\,p-1)$, where p is an odd prime, d is prime. Also: the set G of mappings $x\to\rho^i x+\alpha$ $(i=0, \cdots, d-1)$, where $x\in \mathrm{GF}(p^n)$, $d\,|\,p^n-1$, $H=[G]_{\alpha=0}$.

For the case $\Sigma=(H_1, H_2, H_3)'$ (Σ is two-dimensional) the axioms are more numerous, and the theory more complicated. We cite only theorem 5.1: G has a subgroup H which is its own normalizer. If H and two of its conjugates H_1, H_2 under G have intersection D_1, D_2 respectively, $D_1\neq H$, $D_2\neq H$, then D_1, D_2 are conjugate under H. If D_1, D_2 are distinct, they do not intersect. *J. L. Brenner* (Palo Alto, Calif.)

4973:

Zacher, Giovanni. Un'osservazione sui gruppi finiti p-risolubili. Rend. Accad. Sci. Fis. Mat. Napoli (4) **25** (1958), 46–48.

The author extends to p-solvable groups a theorem given for solvable groups by T. Ikuta [On series of maximal non-normal subgroups in a solvable group, Nat. Sci. Rep. Lib. Arts Fac. Shizuoka Univ. no. 1 (1950)]. The result states that a finite p-solvable group G contains a chain $G=H_0\supset H_1\supset\cdots\supset H_s\supset\cdots\supset H_t$ of subgroups, t the number of p-factors in a principal series for G, such that (1) H_i is maximal in H_{i-1} $(i=1, 2, \cdots, t)$, (2) the indices $H_i{:}H_{i+1}$ $(i=0, 1, \cdots, s-1)$, coincide in some order with the orders of the eccentric principal p-factors of G, (3) H_i is non-normal in H_{i-1} $(i=1, 2, \cdots, s)$, (4) $H_j{:}H_{j+1}=p$ for $j>s$, (5) H_j is normal in H_s for $j>s$, and H_t has order prime to p. A principal p-factor M/N is called eccentric if it is not contained in the center of G/N. *D. G. Higman* (Ann Arbor, Mich.)

4974:

Livčak, Ya. B. A locally solvable group which is not an

RN^*-**group.** Dokl. Akad. Nauk SSSR **125** (1959), 266–268. (Russian)

A group is an RN^*-group if it has an ascending well-ordered series with each factor normal in the following and with abelian factor-groups. In their survey paper "Soluble and nilpotent groups" Kuroš and Černikov [Uspehi Mat. Nauk (N.S.) **2** (1947), no. 3 (19), 18–59; Amer. Math. Soc. Transl. no. 80 (1953); MR **10**, 677; **14**, 618] raised the problem whether a locally soluble group is necessarily an RN^*-group. The answer is in the negative: in the present paper the author constructs a group G which is locally soluble, but not an RN^*-group. In fact, G is semisimple, i.e., it has no locally nilpotent normal subgroup $\neq 1$. The group G also provides negative answers to a number of other questions raised by Kuroš and Černikov [loc. cit.] and more recently by Plotkin [ibid. **13** (1958), no. 4 (82), 89–172; MR **21** #686]. For example, G is locally radical, but not radical. (A group is called radical if it has an ascending normal series with locally nilpotent factors.)

The group G is constructed, so to speak, as the inverse limit of the repeated wreath product of infinite cyclic groups. Let Z be infinite cyclic, $G_1 = Z$, $G_2 = Z \wr G_1$, $\cdots$, $G_{n+1} = Z \wr G_n$. Then $G_1 < G_2 < \cdots < G_n < \cdots$ and $G = \bigcup_{n=1}^{\infty} G_n$. Once the group has been constructed, it is fairly easy to read off all the required unpleasant properties.

K. A. Hirsch (London)

4975:

Kargapolov, M. I. On the theory of $\bar{Z}$-groups. Dokl. Akad. Nauk SSSR **125** (1959), 255–257. (Russian)

A group is called a $\bar{Z}$-group if every homomorphic image of it has a central system. In their survey paper [op. cit., preceding review], Kuroš and Černikov raised the following problems: Is the direct product of an arbitrary set of $\bar{Z}$-groups a $\bar{Z}$-group? Is every subgroup of a $\bar{Z}$-group also a $\bar{Z}$-group? In the present note the author proves that the first problem has an affirmative answer, the second a negative answer. A counter-example is provided by the multiplicative group of 2×2 matrices of the form $\begin{pmatrix}1 & 0\\ r & s\end{pmatrix}$, where r and s are rational, $s \neq 0$ and with an odd denominator. The proof that this group has the property $\bar{Z}$ is not difficult. On the other hand the matrices $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ generate an infinite dihedral subgroup which has finite non-nilpotent homomorphic images. It also shows (this was another problem of Kuroš and Černikov) that a $\bar{Z}$-group need not be an $\tilde{N}$-group, a group being $\tilde{N}$ if through every subgroup of it a normal system can be laid.

K. A. Hirsch (London)

4976:

Eremin, I. I. Groups with finite classes of conjugate abelian subgroups. Mat. Sb. (N.S.) **47** (**89**) (1959), 45–54. (Russian)

Let G be a group which for any of its subgroups has only a finite number of conjugates. B. N. Neumann [Math. Z. **63** (1955), 76–96; MR **17**, 234] has shown that such a group is a finite extension of its center and conversely. The main result of the present paper is a proof of a conjecture of S. N. Černikov according to which the weaker assumption that the abelian subgroups of G have finite classes of conjugate subgroups leads to the same class of groups. There is also determined the class of those torsion groups in which to any abelian p-subgroup there belongs a finite class of conjugate subgroups.

A. Kertész (Debrecen)

4977:

Neumann, B. H. Isomorphism of Sylow subgroups of infinite groups. Math. Scand. **6** (1958), 299–307.

Let an FC-group be a group in which each element has at most a finite number of conjugates; see the author's paper, Proc. London Math. Soc. (3) **1** (1951), 178–187 [MR **13**, 316]. It is known that (A) two Sylow p-subgroups (subgroups maximal with respect to the property of being p-subgroups) S and S' of a periodic FC-group G are isomorphic under an automorphism of G which coincides with a (perhaps different) inner automorphism on each finite subset of G (a locally inner automorphism) [R. Baer, Duke Math. J. **6** (1940), 598–614; MR **2**, 2; P. A. Gol'berg, Mat. Sb. (N.S.) **19** (**61**) (1946), 451–460; MR **8**, 367]. It is also known that (B) if G is a group in which the periodic elements each have finitely many conjugates (a PFC group) then for Sylow p-subgroups S and S', there is an automorphism on the periodic portion of G (which happens to be a subgroup of G) which carries S onto S'. By considering the inverse limit of a directed set of sets of automorphisms, it is shown that if S and S' are Sylow p-subgroups of an FC-group G, then there exists a locally inner automorphism of G carrying S onto S', thus extending the result (A) by removing the periodicity restriction. By considering the (restricted) direct sum of an infinite number of copies of the alternating group of degree 4, and by extending this sum to a PFC group with an outer automorphism of order 2, the author shows that one cannot effect, in general, the isomorphism of (B) on S onto S' by any automorphism of G.

F. Haimo (St. Louis, Mo.)

4978:

Maurer, I. Sur la décomposition des cycles. Gaz. Mat. Fiz. Ser. A **10** (**63**) (1958), 656–657. (Romanian. French and Russian summaries)

"L'auteur démontre qu'un cycle arbitraire (fini ou infini) peut être obtenu comme un produit de deux permutations du second ordre." (From the author's summary)

B. Harris (Evanston, Ill.)

4979:

★**Bergström, Harald. Über gewisse Invarianten von Permutationsgruppen.** Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 38–44. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

If a permutation P of $1, 2, \cdots, n$ replaces i by k_i, put $d(P) = \sum (i-k_i)^2/n$. The mean of $d(P)$ over a transitive subgroup g of the symmetric group γ_n is

$$E_g[d(P)] = \frac{1}{\operatorname{ord}(g)} \sum_g d(P) = E_{\gamma_n}[d(P)] = (n^2-1)/6.$$

If a permutation P of $x_1, \cdots, x_n$ replaces x_i by x_i^P, put $d(x, P) = \sum (x_i - x_i^P)^2/n$. For a transitive subgroup g of γ_n,

$$E_g[d(x, P)] = E_{\gamma_n}[d(x, P)] = \frac{1}{n^2} \sum_i \sum_j (x_i - x_j)^2.$$

For a doubly transitive subgroup

$$E_g[d^2(x, P)] = E_{\gamma_n}[d^2(x, P)]$$
$$= \frac{1}{n^3(n-1)} \sum_{i\neq j} \sum_{\nu\neq\lambda} (x_i - x_\nu)^2 (x_j - x_\lambda)^2$$
$$+ \frac{1}{n^3} \sum_i \sum_\nu (x_i - x_\nu)^4.$$

The variance of $d(x, P)$ over g is $E_g[d^2(x, P)] - E_g^2[d(x, P)]$. For a group only singly transitive the variance is independent.

If $f(x) = f(x_1, \cdots, x_n)$ is any polynomial and $f(x^P)$ is the same polynomial of permuted variables by a permutation P, $\bar{f}_g(x)$ denotes the mean value of $f(x^P)$ over the permutations P of a group g. Then $\bar{f}_{g_1}(x) = \bar{f}_{g_2}(x)$ if and only if $\bar{f}_{g_1}(x)$ is unaltered by permutations of g_2 and conversely.

The module of linear functions of $f(x^P)$ with rational coefficients for permutation P of a group g determines a matrix representation $\vartheta(g)$ of g. The reduction of $\vartheta(g)$ into irreducible representations contains the unit representation either once or not at all, according as $\bar{f}_g(x) \neq 0$ or $\bar{f}_g(x) = 0$. If $\vartheta(g)$ induces a representation $\vartheta_1(g_1)$ in a subgroup g_1 of g, then both $\vartheta(g)$ and $\vartheta_1(g_1)$ contain the unit representation exactly once if and only if

$$\bar{f}_{g_1}(x) = \bar{f}_g(x) \neq 0.$$

D. E. Littlewood (Bangor)

4980:

Bauer, Richard; and Feit, Walter. On the number of irreducible characters of finite groups in a given block. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 361–365.

The blocks of irreducible characters of a finite group with regard to a fixed prime p and their significance for the arithmetic of the group ring have been discussed in several previous publications [Brauer, same Proc. **30** (1944), 109–124; **32** (1946), 182–186, 215–219; MR **6**, 34; 8, 14, 131]. In particular it was stated that the number m of ordinary irreducible characters in a p-block of defect d satisfies $m \leqq p^{d(d+1)/2}$. The authors here prove $m \leqq \frac{1}{4}p^{2d} + 1$. If $\nu(x)$ is the index of the greatest power of p which divides x, g is the order of the group and x_i the degree of a character of a block of defect d,

$$\nu(x_i) = \nu(g) - d + \lambda_i,$$

where $\lambda_i \geqq 0$ is called the height of the character.

The authors go on to obtain such results as the following. If $d \geqq 2$, then $\lambda_i \leqq d-2$; if $d \leqq 2$ then $\lambda_i = 0$ for each character. If $\lambda_i > 0$ for some character then $m < \frac{1}{2}p^{2d-2}$.

D. E. Littlewood (Bangor)

4981:

★Tits, Jacques. Les "formes réelles" des groupes de type E_6. Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 162, 15 pp. Secrétariat mathématique, Paris, 1958. 189 pp. (mimeographed)

Over a given field K, the author defines a "plane" π, its projectivities, polarities, etc., in a manner too complicated to be described here [cf. J. Math. Pures Appl. (9) **36** (1957), 17–38; MR **19**, 44]. He states that the group of projectivities of π is isomorphic to the group of type E_6 given by Chevalley [Tôhoku Math. J. (2) **7** (1955), 14–66; MR **17**, 457], and describes automorphisms of that group geometrically. The main point of the paper is that the group Γ of "glissements" of π which commute with a fixed polarity is simple. The order of Γ for finite K is computed. It seems to the reviewer that Γ coincides with the group $E_6{}^1$ given recently by R. Steinberg from a different point of view [Pacific J. Math. **9** (1959), 875–891].

R. Ree (New York, N.Y.)

4982:

Conrad, Paul. Methods of ordering a vector space. J. Indian Math. Soc. (N.S.) **22** (1958), 1–25.

"Ordonné" voudra dire ici "totalement ordonné". Soient k un corps ordonné (commutatif ou non) et E un espace vectoriel sur k. Par ordre de E nous entendrons un ordre qui fasse de E un espace vectoriel ordonné. L'auteur étudie dans cet article différentes manières (dont l'énoncé déborderait le cadre de ce résumé) de définir un ordre sur E.

E étant ordonné, on appelle composante de E tout espace vectoriel de la forme A/B où A et B sont des sous-espaces vectoriels convexes (c'est-à-dire isolés) de E tels que $B \subset A$, $B \neq A$ et tels qu'il n'y ait pas de sous-espace vectoriel convexe intermédiaire entre A et B. Parmi les résultats intéressants démontrés ici citons les suivants. Pour que, quel que soit l'ordre de E, toutes les composantes de E aient pour dimension 1, il faut et il suffit que E ait pour dimension 1 ou que k soit le corps ordonné des nombres réels; dans le cas contraire on peut toujours ordonner E de sorte que toutes ses composantes aient une même dimension > 1. Pour que l'on puisse ordonner E de sorte qu'il n'ait pas de sous-espace convexe propre, il faut et il suffit que l'une des deux conditions suivantes soit vérifiée: (1) k n'est pas archimédien; (2) k est un sous-corps ordonné du corps R des nombres réels et la dimension de E est inférieure ou égale à la dimension de R sur k.

P. Jaffard (Lyon)

4983:

Conrad, Paul. On ordered vector spaces. J. Indian Math. Soc. (N.S.) **22** (1958), 27–32.

Dans cet article qui fait suite au précédent l'auteur détermine tous les ordres de E dans le cas où E a la dimension 2.

P. Jaffard (Lyon)

4984:

Lyapin, E. S. Inversion of elements in semi-groups. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 65–74. (Russian)

Let A be any semigroup. If $x \in A$ and $xA = A$, then x is said to be right invertible. If $Ax = A$, then x is said to be left invertible. If x is both right and left invertible, then x is said to be 2-sided invertible. Let A_1 be the set of 2-sided invertible elements of A, A_l the set of left invertible elements not in A_1, A_r the set of right invertible elements not in A_1, and A_0 the complement of $A_1 \cup A_l \cup A_r$ in A. If A_1 is non-void, then A has a 2-sided unit. The main result of this note is a complete description of the inclusions $A_iA_j \subset A_k$ ($i, j, k = 1, l, r, 0$) that obtain in all semigroups. The following table presents all of the possible inclusions: in every case, the inclusion given is the best possible. (To read the table, multiply the row on the left and the column on the right.)

$\subset$	1	l	r	0
1	1	l	r	0
l	l	l	0	0
r	r	A	r	$r\cup 0$
0	0	$l\cup 0$	0	0

An element $x\in A$ is called left [right] increasing if for some proper subset B of A, $xB=A$ [$Bx=A$]. It is also shown that no element can be both left and right increasing. See also E. S. Lyapin [Mat. Sb. (N.S.) **30** (**80**) (1956), 373–388; Leningrad. Gos. Ped. Inst. Uč. Zap. **89** (1953), 55–65; MR **17**, 825, 942]. *E. Hewitt* (Seattle, Wash.)

4985:

Sagastume Berra, A. E. Theory of ova. Math. Notae **16** (1957/58), 43–59, 65–77. (Spanish)

An ovum is a finite commutative semi-group. Two elements are said to be associated if they are multiples of one another. The author investigates this relation. An ovum in which no distinct elements are associated is a holoid. All holoids of order up to six are found.

H. A. Thurston (Vancouver, B.C.)

4986:

Maury, Guy. Une caracterisation des demi-groupes noethériens intégralement clos. C. R. Acad. Sci. Paris **248** (1959), 3260–3261.

This note is an extension of an earlier note of the author [same C. R. **247** (1958), 254–255; MR **20** #6474]. A characterization, based on the earlier results, of integrally closed noetherian semigroups is obtained which is the analogue of a result for noetherian rings obtained by M. Yoshida and M. Sakuma [J. Sci. Hiroshima Univ. Ser. A **17** (1954), 311–315; MR **16**, 560].

G. B. Preston (Shrivenham)

4987:

Sade, Albert. Quasigroupes obéissant à certaines lois. Rev. Fac. Sci. Univ. Istanbul. Sér. A **22** (1957), 151–184. (Turkish summary)

This paper is stated to have a direct connection with balanced block designs; but this connection is not very obvious, nor made much use of. After listing more than 60 restrictive assumptions, some of them identical relations, others identical implications, yet others of more complicated nature, which can be imposed on quasigroups, the implications and interrelations of a few of them are studied. Quasigroups are here defined as a class of groupoids, so that they are not equationally defined. The quasigroups in which every pair of distinct elements generates a subquasigroup of fixed order k are investigated in some detail (they are the quasigroups that are perhaps most closely related to balanced incomplete block designs). For $k=3$ they correspond to the Steiner triple systems and can be characterized by the (redundant) set of laws (1) $xx=x$, (3) $xy=yx$, (4) $(xy)y=x$, (5) $x(xy)=y$. The quasigroups subject to just (4), or to (4) and (5) (and consequently also (3)) are further studied, especially also with respect to their isotopes of the same kind. There is a bibliography of 70 items and an index. Numerous misprints, most of them obvious, were noted.

B. H. Neumann (Manchester)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 5091, 5160, 5197.

4988:

Maurer, I. Über die Normalteiler der topologisierten Permutationsgruppen. Com. Acad. R. P. Romîne **8** (1958), 5–11. (Romanian. Russian and German summaries)

G denotes the group of all permutations (one-one mappings onto itself) of an arbitrary infinite set M of power $\aleph_\mu$. It was shown by R. Baer (Studia Math. **5** (1934), 15–17] that the normal subgroups of G form the composition series

$$\{\varepsilon\}\subset P\subset G_0\subset G_1\subset\cdots\subset G_\nu\subset\cdots\subset G_\mu\subset G.$$

Here, ν being any ordinal number $(0\leq\nu\leq\mu)$, G_ν denotes the set of permutations $\pi(\xi)$ $(\xi\in M)$ for which $\pi(\xi)\neq\xi$ is valid only for a subset of M of power less than $\aleph_\nu$. Thus G_0 comprises the finite permutations in G; and P is the subset of even permutations in G_0. The author [review below] has introduced a topology for G, based on a definition of the limit of a sequence $\{\pi_r\}_{r<\omega}$ of permutations. He now determines all the proper normal subgroups of the topological group G: they are those G_ν $(\nu\neq 0)$ whose indices ν are ordinal numbers of the first kind, or ordinal numbers of the second kind which are not limits of sequences of type ω. This generalizes the author's earlier result for a denumerable M [Acad. R. P. Romîne Bul. Şti. Secţ. Şti. Mat. Fiz. 8 (1956), 265–272; MR **18**, 907].

I. M. H. Etherington (Edinburgh)

4989:

Maurer, I. Gy. Eine Topologisierung der Permutationsgruppen einer beliebigen unendlichen Menge. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2** (**50**) (1958), 55–59.

This account in German overlaps the author's Romanian paper [review above]. There is one further theorem, namely that the topological group G of all permutations of an arbitrary infinite set M has only inner automorphisms. This result, already proved by the author for the case when M is denumerable [op. cit., review above], is derived from the corresponding theorem for G as an abstract group [J. Schreier and S. Ulam, Fund. Math. **28** (1936), 258–260]. *I. M. H. Etherington* (Edinburgh)

4990:

Coleman, A. J. The Betti numbers of the simple Lie groups. Canad. J. Math. **10** (1958), 349–356.

If G is a compact simple Lie group, its Poincaré polynomial is known to take the form $\prod_1^n (1+t^{2m_i+1})$, where n is the rank of G. The exponents m_i thus completely describe the polynomial and considerable work has gone into determining these numbers. The author starts with a brief historical review and then brings his contribution, which clarifies an empirical observation due to Coxeter, to the effect that if R is the product of the generating reflections across the walls of the fundamental chamber of G then the eigenvalues of R determine these exponents. Starting from the definition of the exponents in terms of the minimal invariant polynomials of the Weyl group,

and using the algebraic structure of the ring of invariant polynomials, due to C. Chevalley, the author derives the Coxeter definition once a relatively simple empirical observation is granted. *R. Bott* (Ann Arbor, Mich.)

4991:

Chen, Kuo-Tsai. Exponential isomorphism for vector spaces and its connection with Lie groups. J. London Math. Soc. **33** (1958), 170–177.

In this paper the author proceeds roughly in the manner outlined below.

Let K be a field, and L a vector space over K. For p a non-negative integer, let $T_p(L)$ be the tensor product of the vector space L with itself p-times, i.e., $T_0(L)=K$, $T_{p+1}(L)=T_p(L)\otimes_K L$ for $p\geq 0$. Let $T(L)$ be the tensor algebra of L. As a vector space $T(L)=\sum_{p\geq 0}T_p(L)$, i.e., $T(L)$ is the direct sum (weak direct sum) of the vector spaces $T_p(L)$. Let $\Lambda(L)=\prod_{p\geq 0}T_p(L)$, i.e., $\Lambda(L)$ is the direct product (strong direct sum) of the vector spaces $T_p(L)$. Recall that if $x=x_1\otimes\cdots\otimes x_p\in T_p(L)$, $y=y_1\otimes\cdots\otimes y_q\in T_q(L)$, then $x\cdot y$ is an element of $T_{p+q}(L)$, and $x\cdot y=x_1\otimes\cdots\otimes x_p\otimes y_1\otimes\cdots\otimes y_q$. The multiplication in $T(L)$ extends naturally so as to make $\Lambda(L)$ into an algebra.

Now proceed to define two new algebras over K. Let $S_p(L)=T_p(L)$, $S(L)=\sum_{p\geq 0}S_p(L)$. For p, q non-negative integers let (p,q) be the quotient of $(p+q)!$ by $p!q!$. Make $S(L)$ into an algebra over K as follows: Suppose $x=x_1\otimes\cdots\otimes x_p\in S_p(L)$, and $y=y_1\otimes\cdots\otimes\in S_q(L)$; then $x\cdot y\in S_{p+q}(L)$ and $x\cdot y=(p,q)x_1\otimes\cdots\otimes x_p\otimes y_1\otimes\cdots\otimes y_q$. Now one checks without difficulty that $S(L)$ is an algebra over K. Let $\Gamma(L)=\prod_{p\geq 0}S_p(L)$ and extend the multiplication in $S(L)$ to $\Gamma(L)$ so that $\Gamma(L)$ becomes an algebra with subalgebra $S(L)$. An element of $\Gamma(L)$ is an infinite sequence of elements y_n such that $y_n\in S_n(L)$. Such an infinite sequence will be denoted by $\sum_{n=0}^\infty y_n$. If $x=\sum_{n=0}^\infty x_n$, $y=\sum_{n=0}^\infty x_n$ then $xy=\sum_{n=0}^\infty z_n$ where $z_n=\sum_{i+j=n}(i,j)x_iy_j$. Suppose $a\in L$, and that $x_0=1$, and x_n is the tensor product of a with itself n times. Denote by $\gamma(a)$ the element $\sum_{n=0}^\infty x_n$. Observe that $\gamma(a)$ is a unit. In fact $\gamma(a)(\gamma(-a))=1$.

Let $M(L)$ be the free monoid generated by the elements of L, and let $\alpha: M(L)\to\Gamma(L)$ be the multiplicative map such that $\alpha(a)=\gamma(a)$ for $a\in L$. Denote by $G(L)$ the image of $M(L)$ in $\Gamma(L)$. Observe that $G(L)$ is a group: $\alpha(0)=1\in G(L)$, and $\alpha(a_1\cdots a_p)^{-1}=\alpha((-a_p)\cdots(-a_1))$, where $a_1,\cdots,a_p\in L$. Denote by $K(G(L))$ the group algebra of $G(L)$ over K. Define $\bar{\alpha}$: $K(G(L))\to\Gamma(L)$ to be the map of K-algebras such that if $x\in M(L)$, then $\bar{\alpha}(\alpha(x))=\alpha(x)$. The first theorem of the author asserts that if K is a field of characteristic zero then $\bar{\alpha}$ is a monomorphism.

If K is of characteristic zero, define ξ: $\Gamma(L)\to\Lambda(L)$ by $\xi(\sum_{n=0}^\infty x_p)=\sum_{n=0}^\infty(1/p!)x_p$. The map ξ is an isomorphism of K-algebras. Let $\theta=\xi\bar{\alpha}$: $K(G(L))\to\Lambda(L)$. The author calls the map of K-algebras θ, the exponential map, and has that it is a monomorphism.

Suppose G is a Lie group (real or complex), and that K is the field of definition of G. Let L be the Lie algebra of G, and $\exp: L\to G$ the classical exponential map. Theorem: There is a unique map ψ: $G(L)\to G$ such that $\psi(\alpha(a))=\exp a$ for $a\in L$, and (1) ψ is a homomorphism of groups, and (2) the image of ψ is the component of the identity in G. *J. C. Moore* (Princeton, N.J.)

4992:

Chen, Kuo-Tsai. Integration of paths—a faithful representation of paths by non-commutative formal power series. Trans. Amer. Math. Soc. **89** (1958), 395–407.

In this review no attempt will be made to describe the methods used by the author in proving his theorems, nor will there be any consideration of differentiability conditions or other smoothness conditions in spite of their importance.

Let R^m denote Euclidean m-space, $R=R^1$. Let $f_1,\cdots,f_p:[a,b]\to R$ be piecewise continuous functions on $[a,b]=\{t\mid t\in R,\ a\leq t\leq b\}$. Let $g_1(t)=\int_a^t f_1(s)ds$, $g_{i+1}(t)=\int_a^t g_i(s)f_{i+1}(s)ds$ for $i=1,\cdots,p-1$. Denote $g_p(b)$ by $\int_a^b f_1\cdots f_p$.

Suppose M is an m-dimensional manifold, $\omega_1,\cdots,\omega_m$ are 1-forms on M. Let α be a curve in M, i.e., for some a, $b\in R$, $\alpha:[a,b]\to M$. Let $\lambda_1,\cdots,\lambda_m$ be the induced 1-forms on $[a,b]$. These may be considered as real-valued functions. Let $\int_\alpha\omega_1\cdots\omega_p=\int_a^b\lambda_1\cdots\lambda_p$. Define $\theta(\alpha)$ to be a formal power series in the non-commutative indeterminates $X_1,\cdots,X_m$ as follows:

$$\theta(\alpha)=1+\sum_{p=1}^{\infty}\sum\left(\int_\alpha\omega_{i_1}\cdots\omega_{i_p}X_{i_1}\cdots X_{i_p}\right).$$

If a, $b\in R$, $a<b$, then $\lambda:[a,b]\to[a,b]$ is a change of parameter if and only if λ is continuous, and (1) $\lambda(a)=a$, (2) $\lambda(b)=b$, (3) if $r<s$, r, $s\in[a,b]$, then $\lambda(r)<\lambda(s)$.

Theorem 1: Let $dx_1,\cdots,dx_n$ be the canonical 1-forms on R^n. If α, $\beta:[a,b]\to R^n$ are sufficiently smooth paths, then $\theta(\alpha)=\theta(\beta)$ if and only if there exists a translation T of R^n, and a change of parameter λ: $[a,b]\to[a,b]$ such that $\alpha=T\beta\lambda$.

Theorem 2: Let G be a Lie group of dimension n, and let $\omega_1,\cdots,\omega_n$ be a basis for the left invariant 1-forms on G. If α, β: $[a,b]\to G$ are sufficiently smooth paths, then $\theta(\alpha)=\theta(\beta)$ if and only if there exists a left translation T of G and a change of parameter λ: $[a,b]\to[a,b]$ such that $\alpha=T\beta\lambda$. *J. C. Moore* (Princeton, N.J.)

4993:

Hewitt, Edwin. The asymmetry of certain algebras of Fourier-Stieltjes transforms. Michigan Math. J. **5** (1958), 149–158.

Soit G un groupe abélien localment compact, satisfaisant la condition (1): tout voisinage de 0 dans G contient un élément d'ordre infini. L'auteur démontre que l'algèbre de convolution $\mathscr{M}(G)$ (des mesures bornées sur G) est asymétrique [si $G=R$, le résultat est dû à Šreider, Mat. Sb. (N.S.) **27** **(69)** (1950), 297–318; MR **12**, 420] et indique une série de corollaires, concernant le spectre de $\mathscr{M}(G)$ (non densité du groupe des caractères de G, etc.). Sur la démonstration, voir la revue suivante. Remarque du référent: La condition (1) est inutile, et le résultat vaut pour tout G discret, comme l'a montré depuis Williamson [Communication au Congrès International des Mathématiciens, Edinburgh 1958; voir aussi Helson, Kahane, Katznelson et Rudin, Acta Math. **102** (1959), 135–157]. *J. P. Kahane* (Montpellier)

4994:

Rudin, Walter. Independent perfect sets in groups. Michigan Math. J. **5** (1958), 159–161.

L'auteur donne une démonstration simple du lemme essentiel utilisé par E. Hewitt dans l'article du même

journal [voir revue précédente]: tout groupe G satisfaisant (1) contient un ensemble E homéomorphe à l'ensemble de Cantor, indépendant dans le sens suivant: pour tout choix de $x_1, x_2, \cdots, x_k$ distincts dans E, et des entiers $n_1, n_2, \cdots, n_k$ non tous nuls, on a $n_1x_1+n_2x_2+\cdots+n_kx_k \neq 0$. *J. P. Kahane* (Montpellier)

4995:

★Наймарк, М. А. Линейные представления группы Лоренца. [Naĭmark, M. A. Linear representations of the Lorentz group.] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 376 pp. 14.15 rubles.

The purpose and contents of this book are well described in an Annotation on the second page. "This book is the first monograph on the representations of the Lorentz group. It is written principally for theoretical physicists, but the principal results and methods (belonging to the author) must also be interesting to specialists in mathematics. The exposition begins with elementary concepts of the theory of groups and the theory of group representations; facts about the Lorentz group and the rotation group in three-dimensional Euclidean space are given in detail, so that the book can serve as a means for studying the general theory of representations."

In theory, anyone who knows elementary analysis could read the book. Every concept needed, and there are many, is defined accurately if briefly. The book will certainly prove useful to mathematicians interested in representations of locally compact noncommutative groups. The number of theoretical physicists who will actually master it may be fairly small. It presents rapidly a great many concepts, and it is filled with computations, some of them forbidding indeed. The book shows signs of hasty proofreading, although its writing was evidently thought out with great care.

Chapters I and II contain only classical material. The 3-dimensional rotation group G_0, the 2×2 unitary group $\mathfrak{U}$, and the Lorentz group $\mathfrak{G}$ ($\mathfrak{G}_+$ denotes the proper Lorentz group) are defined. A complete description is given of the irreducible continuous representations of G_0 and $\mathfrak{U}$ and of the decomposition into irreducible parts of any continuous representation of $\mathfrak{U}$ by bounded operators on a reflexive Banach space R. This last generalizes at once to any compact group.

Chapter III, which is 198 pages long, gives a description of all continuous completely irreducible representations of $\mathfrak{G}$ and $\mathfrak{G}_+$ by linear bounded operators on R. (Complete irreducibility reduces to irreducibility if R is finite-dimensional or the representation is unitary.) It begins with a construction of infinitesimal operators for a certain class of irreducible representations of $\mathfrak{G}_+$ by operators in R. These operators are very like those classically used for obtaining the irreducible representations of G_0, and they are used to construct a certain class of irreducible representations of $\mathfrak{G}_+$ (which later turn out to be all completely irreducible representations). The analogy with representations of G_0 is emphasized. The formulas used here were first published by Harish-Chandra [Proc. Royal Soc. London Ser. A **189** (1947), 372–401; MR **9**, 132]. The author next introduces the group $\mathfrak{A}$ of 2×2 unimodular complex matrices, and uses its homomorphic mapping onto $\mathfrak{G}_+$ to show that the representations of $\mathfrak{G}_+$ previously constructed are in fact all of the completely irreducible representations of $\mathfrak{G}_+$. This is a long job. On the way, he explains the motivation for, and obtains explicitly, the trace for operators appearing in representations of $\mathfrak{A}$, the analogue of Plancherel's formula, and various subalgebras of $L_1(\mathfrak{A})$. Much of the material in this chapter is due to the author and/or I. M. Gel'fand. [See, for example, Gel'fand and Naĭmark, Izv. Akad. Nauk SSSR. Ser. Mat. **11** (1947), 411–504; *Unitarnye predstavleniya klassičeskih grupp*, Trudy Mat. Inst. Steklov. vol. 36, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1950; MR **9**, 495; **13**, 722; Naĭmark, Uspehi Mat. Nauk (N.S.) **9** (1954), no. 4 (62), 19–93; Dokl. Akad. Nauk SSSR **112** (1957), 583–586; MR **16**, 566; **20** #1240]. The chapter closes with an extension of the preceding work to the group $\mathfrak{G}$.

Chapter IV deals with "invariant equations", which for the group $\mathfrak{G}_+$ are equations of the form

$$(*) \qquad \sum_{j=1}^{4} L_j \frac{\partial\psi}{\partial x_j} + i\kappa\psi = 0,$$

where ψ is a function defined on real (x_1, x_2, x_3, x_4)-space, with values in a reflexive Banach space R, the L_j are closed linear operators defined on subspaces of R, and where (*) is invariant when (x_1, x_2, x_3, x_4) undergoes a Lorentz transformation g while ψ undergoes the transformation T_g, where $g\to T_g$ is a representation of $\mathfrak{G}_+$ by operators in R. The general form of such equations is obtained (also for G_0 and $\mathfrak{G}$). A general notion of Lagrangian is introduced, in terms of a bilinear Hermitian form on R invariant under a representation T_g, and is used to obtain certain invariant equations. Beginning on p. 340, applications to physics are discussed.

The book closes with nine short appendices, discussing Jacobians, Burnside's theorem, closed operators, and so on. *E. Hewitt* (Seattle, Wash.)

4996:

Conner, P. E. Orbits of uniform dimension. Michigan Math. J. **6** (1959), 25–32.

Let (L, X) be a compact connected Lie group L acting as a transformation group on a locally compact, arcwise connected, finite dimensional separable metric space X. Denote by H_x the identity component of the isotropy group G_x at x, and F_x the set of points of X fixed under H_x. The group B_x of elements leaving F_x invariant contains the normalizer of H_x. Let $\hat{B}_x = B_x/H_x$, and Q denote the rational field. The principal result and tool of the paper is the following: If X is simply connected, there is a spectral sequence $\{E_2^{s,t}\}$, with

$$E_2^{s,t} \simeq H^s(L/B_x \times X/L; Q) \otimes H^t(\hat{B}_x; Q)$$

whose E_∞-term is associated with $H^*(X; Q)$. A corollary states that if further H_x is of maximal rank in L, then X is topologically the product $L/B_x \times X/L$.

The author uses the principal result and A. Borel's Principal Algebraic Theorem [Ann. of Math. (2) **57** (1953), 115–207; MR **14**, 490] to prove that if X is simply connected and is a rational cohomology sphere, then either X/L is acyclic over the rationals or else L is a circle group or a rational 3-sphere acting with finite isotropy groups. Several other interesting results are proved.

The paper is marred by a number of annoying errors—none of a serious nature, however. In the proof of Theorem 1, line 9 from the bottom, the circumflex is omitted in three places, and in line 12, y should read y'. In Theorem 4, it should be stated that at least one lower

dimensional orbit exists. Also, "maximal" means "maximal rank". The remark added in proof is confusing and apparently unnecessary.

P. S. Mostert (New Orleans, La.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

See 5103, 5151.

FUNCTIONS OF REAL VARIABLES

See also 4889.

4997:

Bogel, Karl; und Brauning, Günter. Über die Darstellbarkeit einer Funktion durch ihre Taylorreihe im Reellen. Wiss. Z. Hochschule Elektrotechn. Ilmenau **3** (1957), 1–3.

If I is a finite interval on the line and U an open set in I, one may construct an infinitely differentiable function which is represented by its Taylor series in $I-U$, but not in U. *D. Waterman* (Lafayette, Ind.)

4998:

Stein, P. A note on extreme values of a function of several variables. Amer. Math. Monthly **66** (1959), 895–896.

The paper is a neat exposition of known conditions for existence of an extremum or a saddle point.

W. Kaplan (Ann Arbor, Mich.)

4999:

Iosifescu, Marius. Propriétés différentielles des fonctions réelles d'une variable réelle, jouissant de la propriété de Darboux. C. R. Acad. Sci. Paris **248** (1959), 1918–1919.

f: fonction réelle finie définie sur l'intervalle (a, b). E_t désigne l' "ensemble de niveau" $\{x; a<x<b, f(x)=t\}$. Un point $x_0\in(a, b)$ est dit isolé unilatéralement ou bilatéralement s'il l'est pour l'ensemble E_t où $t=f(x_0)$. Cinq théorèmes sont énoncés pour les f jouissant de la propriété de Darboux, généralisant des résultats pour la plupart déjà connus pour des f continues. Il s'agit principalement de la répartition des configurations de Denjoy pour le faisceau dérivé de f aux points isolés unilatéralement ou bilateralement. Les démonstrations paraîtront dans un périodique roumain. Une indication est donnée: Le théorème de Lebesgue sur la dérivabilité des fonctions monotones est utilisé au départ. {Remarques du rapporteur: Les théorèmes 1, 2 et 3 s'obtiennent aisément à partir des théorèmes de Roger-Saks [S. Saks, *Theory of the integral*, Stechert, New York, 1937; pp. 264–268] en observant qu'en un point x_0 isolé unilatéralement il existe un intervalle $I=(x_0, x_0+h)$ ou $I=(x_0-h, x_0)$, $h>0$, tel que, pour $x\in I$, $f(x)-f(x_0)$ soit constamment $\geqq 0$ ou constamment $\leqq 0$, donc que le faisceau dérivé laisse échapper les directions d'un quadrant ouvert.} *Chr. Pauc* (Nantes)

5000:

Hoang Tuĭ [Hoang Tuy]. The structure of measurable functions. Dokl. Akad. Nauk SSSR **126** (1959), 37–40. (Russian)

For a measurable function $f(x)$ $(0\leqq x\leqq 1)$, the author defines, at a point x, (semi-) derivates in density c, where $0\leqq c\leqq 1$, analogous to those of density 1 (essential derivates), or what amounts to the same in effect, the claim of a real number t to be such a derivate is assessed by a limiting density $c=c(t)$. Various pathological examples are announced, inspired by work of Plamennov [Mat. Sb. (N.S.) **42** (**84**) (1957), 223–248; MR **19**, 639] and Marcinkiewicz, for instance the following: Let $0<a<1$ and let O, A, B be disjoint sets, defined by a decomposition of the reals t into three sets, where O is open and A is a countable intersection of opens; then there exists a continuous $f(x)$ $(0\leqq x\leqq 1)$ such that, at almost each x, no value of t satisfies $0<c(t)<a$, and O, A, B are the sets of t in which $c(t)$ is, respectively, 0, a, or greater.

L. C. Young (Madison, Wis.)

MEASURE AND INTEGRATION

5001:

Phakadze, Š. S. The theory of Lebesgue measure. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **25** (1958), 3–271. (Russian)

The work is devoted to some of the deeper problems of measurability in Euclidean n-space. The group I of congruences plays an important part and invariance is understood to be relative to it. Further generalizations are obtained by replacing it by a subgroup F, and occasionally by an arbitrary group of one-to-one mappings of n-space onto itself. An invariant additive class of sets which includes the unit cube as a member is termed admissible, and a measure on an admissible class is termed an I-measure if it takes a finite value for the unit cube, or in particular the value unity (normalized I-measure). An admissible class is soluble if an I-measure exists, complete if further the subclass of its sets of I-measure zero is then hereditary. Borel sets, and measurable sets in the sense of Lebesgue, constitute minimal soluble and complete classes, respectively. The author defines further a number of fundamental notions, notably: absolute nullsets, properly almost invariant sets; the latter are of four types, one depending on an arbitrary ordinal. We omit their definitions here.

Considerable space is devoted to examples of nonmeasurable properly almost invariant sets and nonmeasurable absolute nullsets, the latter only in the cases $n=1, 2$; it is verified that a countable union of absolute nullsets may fail to be an absolute nullset. An appendix provides further an effective construction of a continuous subgroup, of measure zero, of the reals, which has continuum-many co-sets.

The problem of the existence of an I-measure in an admissible class is equivalent to that of extending Lebesgue measure from the corresponding minimal class. Two special types of I-measures are introduced in this connection: type A, for which every admissible set differs by I-measure zero from a Lebesgue measurable set, and type B, still more important, for which, corresponding to each admissible set, there is a countable intersection M of sets congruent to it, such that M has I-measure zero. It is observed that, for an I-measure not of type B, the admissible class of sets has continuum-many other normalized I-measures.

Naturally much of the work has a bearing on the continuum hypothesis, or on its weaker form according to which no inaccessible cardinal can be less than, or equal to, the cardinal of the continuum. Thus the existence of an I-measure, which is equivalent to that of an extension of an I-measure from the minimal class where it coincides with Lebesgue measure, leads to the question, raised by Sierpiński, of the existence of a non-extendable I-measure; and the answer to the latter is negative, subject to the weaker continuum hypothesis. On the other hand, it is shown that it is possible to extend any soluble class and any I-measure of type B. The continuum hypothesis itself is shown to be equivalent to the existence of a set E, of Lebesgue measure zero, which is properly almost invariant in the appropriate sense.

The final chapter studies the structure of I-measures not of type B and establishes for them a canonical decomposition in terms of a residual partial measure and a maximal transfinite sum of element-measures, without settling, however, the question whether the residual partial measure must vanish identically. There is a corresponding transfinite decomposition of n-space into sets, where the ordinal of the decomposition is the first ordinal whose cardinal is that of the continuum. A variety of such decompositions, with this or a smaller ordinal, occur in the work; for instance, the constituent sets can be taken to be properly almost invariant, and one of the relevant interpretations then concerns the existence of continuum-many solutions x, y of the equation $x+y=z$ where x, y are restricted to belong to a set or to its complement. [Cf. Erdös, Michigan Math. J. **2** (1953), 51–57; MR **16**, 20.] *L. C. Young* (Madison, Wis.)

5002:

Varadarajan, V. S. A remark on strong measurability. Sankhyā **20** (1958), 219–220.

Es sei $(\Omega, \mathscr{S})$ ein meßbarer Raum, in dem die sigma-Mengenalgebra $\mathscr{S}$ höchstens die Mächtigkeit c des Kontinuums hat, und ϕ eine meßbare Abbildung von Ω in einen metrischen Raum X. Unter der Hypothese $2^{\aleph_1}>c$ wird bewiesen, daß $\varphi(\Omega)$ separabel und daher φ stark meßbar ist. *K. Krickeberg* (Aarhus)

5003:

Seregin, L. V. Stationary measures in a space of sequences. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 151–154. (Russian)

Let Ω_N denote the set of all doubly infinite sequences $s=\cdots s_{-1}, s_0, s_1, \cdots$ in the set $\{0, 1, \cdots, N-1\}$, and let π denote the shift transformation for which always $(\pi s)_k=s_{k-1}$. A cylinder in Ω_N is a set of the form $\{i_{-m}\cdots\hat{i}_0\cdots i_n\}$, consisting of all $s\in\Omega_N$ for which $s_k=i_k$ whenever $-m\leqq k\leqq n$. (The circumflex indicates the element numbered 0.) Let $\mathfrak{A}_N$ denote the σ-algebra generated by the cylinders in Ω_N. A measure μ on $\mathfrak{A}_N$ is called stationary provided $\mu(\pi A)=\mu A$ for all $A\in\mathfrak{A}_N$, and is ergodic provided, whenever $A\in\mathfrak{A}_N$ and $\pi A=A$, $\mu A=0$ or $\mu(\Omega_N\sim A)=0$. M_N denotes the set of stationary measures. Let the subset Q_N of Ω_2 be defined by

$$Q_N=\bigcup_{n=-\infty}^{n=+\infty}\pi^n\{\widehat{1}1\cdots1\},$$

where the braces contain N 1's. The author produces a biunique linear transformation α of M_N onto the set of all $\nu\in M_2$ for which $\nu Q_N=0$, such that $\mu\in M_N$ is ergodic if and only if $\alpha(\mu)\in M_2$ is ergodic, and such that weak convergence in M_N corresponds under the transformation α to that in $\alpha(M_N)\subset M_2$. (The sequence μ_n is said to be weakly convergent to the measure $\mu_0\in M_N$ provided $\lim_{n\to\infty}\int_{\Omega_N}\varphi d\mu_n=\int_{\Omega_N}\varphi d\mu_0$ for every bounded continuous real function φ on Ω_N.) Also included is an extension to the space Ω_∞. *V. L. Klee, Jr.* (Copenhagen)

5004:

Zamansky, Marc. Groupes de Riesz et théories de l'intégration. C. R. Acad. Sci. Paris **248** (1959), 3393–3395; erratum, **249** (1959), 1998.

The completion of an l-group by Cauchy sequences is used to construct by completion Daniell integrals. Let A be a set and let E be an l-group of real-valued functions defined on A. Let I be an o-homomorphism of E into R. I induces a pseudo-topology on E. Let L be the set of all functions from A to $\bar{R}$ that are limits of Cauchy sequences almost everywhere. Let L' be the quotient of L with respect to the equivalence relation $f\equiv g$ almost everywhere. Then for suitable restrictions on E and A there is a faithful representation of the completion E' of E in L'. This representation is used to construct integrals. *P. F. Conrad* (New Orleans, La.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 4889, 4941, 4950, 4954, 5126, 5127, 5205.

5005:

Clunie, J. The minimum modulus of a polynomial on the unit circle. Quart. J. Math. Oxford Ser. (2) **10** (1959), 95–98.

Let $P_n(z)=\sum_0^n a_\nu z^\nu$ with $|a_\nu|\leqq 1$, and denote by $m(P_n)$ the minimum of $|P_n(z)|$ on the circle $|z|=1$. By taking the average of $|P_n(z)|^2$ on $|z|=1$, one concludes easily that $m(P_n)\leqq(n+1)^{1/2}$, and the question was raised (by P. Erdös) how large $m(P_n)$ can become. The author shows that the above-mentioned upper estimate for $m(P_n)$ is essentially the best possible result. For every n, there exists a polynomial $Q_n(z)$ whose coefficients do not exceed 1 in modulus and for which $m(Q_n)\geqq An^{1/2}$, where A is a universal constant. *F. Herzog* (East Lansing, Mich.)

5006:

Parodi, Maurice. Sur quatre méthodes d'étude des zéros des polynomes. Bull. Sci. Math. (2) **82** (1958), 106–117.

After writing a given polynomial

$$f(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n \quad (a_n\neq 0)$$

in several different determinant forms, the author applies Hadamard's condition for the non-vanishing of a determinant and thus obtains results such as the following. All the zeros of $f(z)$ lie in the union of the two circular regions

$$C_1:|z|\leqq n-1,$$
$$C_2:(n-1)!|z-a_1|\leqq(n-2)!|a_2|+\cdots+|a_n|;$$

if these regions are disjoint, exactly one zero lies in C_2. All the zeros of the lacunary polynomial $f(z)=z^n+a_pz^{n-p}+\cdots+a_{n-1}z+a_n$ lie in the intersection of the circular region $p|z|\leq p+1$ and the region

$$|z^p+a_p| \leq p[|a_n/n|+|a_{n-1}/(n-1)|+\cdots+|a_{p+1}/(p+1)|].$$

No references are given to previous results either of the author or of others. *M. Marden* (Milwaukee, Wis.)

5007:

Marden, Morris. On the zeros of infrapolynomials for partly arbitrary point sets. Proc. Amer. Math. Soc. **10** (1959), 391–394.

Let $p(z)=z^n+a_1z^{n-1}+\cdots$ be a polynomial. Some well-known results concern the location of some of the zeros of $p'(z)$ when the positions of only some of the zeros of $p(z)$ are prescribed. The author proposes to develop analogous results for infrapolynomials. He states his main result as follows. Let $p(z)$ be an infrapolynomial on the set $S=S_0+S_1$ where S_0 is a compact point set (finite or infinite) and S_1 is a set of k points, $0\leq k\leq n$. Let T_0 be the set comprised of all points from which S_0 subtends an angle of at least $\pi/(k+1)$. If $p(z)\neq 0$ on S, then $p(z)$ has at most k zeros outside T_0 irrespective of the location of S_1. The author remarks that the special case $k=0$ of his theorem is due to Fejér. *A. Edrei* (Syracuse, N.Y.)

5008:

★Specht, W. Algebraische Gleichungen mit reellen oder komplexen Koeffizienten. Enzyklopädie der mathematischen Wissenschaften: Mit Einschluss ihrer Anwendungen, Bd. I, 1, Heft 3, Teil II. B. G. Teubner Verlagsgesellschaft, Stuttgart, 1958. 76 pp. DM 19.00.

The author reports on the subject called "Analytic theory of polynomials", i.e. the properties of polynomials considered as special analytic functions of one complex variable, centering especially on the dependence of the zeros of a polynomial on its coefficients, or on the dependence of the zeros of polynomials of the type $F(z, f(z), \cdots, f^{(k)}(z))$ (where F is a given polynomial in $k+2$ variables) on the zeros of f. A short summary of the book follows. (A) Existence of roots (the "fundamental theorem of algebra" and its various proofs). Symmetric functions of the roots, continuity of the roots as functions of the coefficients, and their (local) analytic expression as such. (B) Majorations of the roots by various simple functions of the coefficients. The Landau-Montel problem: find a majoration for p roots of the polynomial $1+a_1z+\cdots$ when only $a_1, \cdots, a_p$ are given. (C) Computation of the number of roots of a given polynomial in a given set Γ: the Sturm theorem when Γ is an interval of the real line and the polynomial has real coefficients, the theorems of Cohn and I. Schur when Γ is a disc, the Hurwitz criterion when Γ is a half-plane, and finally the theorems of E. Schmidt and Turán-Erdös which give a majoration of the number of roots in a given angular sector. (D) Position of the roots of the derivative: the Gauss-Lucas theorem and its various generalizations. (E) "Composition theorems": the theorem of Grace and similar results giving information on the roots of

$$\sum_{p=0}^{n}\binom{n}{p}a_pb_pz^p$$

when the roots of

$$\sum_{p=0}^{n}\binom{n}{p}a_pz^p \quad\text{and}\quad \sum_{p=0}^{n}\binom{n}{p}b_pz^p$$

are in given domains. There is no attempt at complete coverage of the relevant literature, but the general methods are clearly delineated and the booklet is a valuable adjunction to the monographs on the subject. *J. Dieudonné* (Paris)

5009:

Meĭman, N. N. On the theory of functions of classes HB and B. Dokl. Akad. Nauk SSSR **125** (1959), 974–977. (Russian)

The author obtains various properties of functions of classes HB and B in general domains; some of them are applied in the paper reviewed below. For terminology, see his previous papers, especially same Dokl. **120** (1958), 1191–1193; **124** (1959), 1211–1214 [MR **20** #5860; **21** #2732]. The first theorem gives conditions for $\omega(z)-tf(z)$ to belong to class B and have the same number of zeros as ω when ω and f are admissible in the same region and f is "smaller" than ω on the boundary. The main part of the paper depends on the following construction of functions belonging to classes HB and B in the regions that the author calls of type $\mathfrak{G}_\beta$. Let γ be a system of open real intervals $\gamma^\nu=(x_{-1}^\nu, x_{+1}^\nu)$. Let $\omega(z)=u(z)+iv(z)$ belong to HB in the upper half-plane, with $v(z)$ having at least one zero x_0^ν in each γ^ν and $u(x)\neq 0$ at the endpoints of the γ^ν. The complementary intervals form the system of cuts β. The function

$$\gamma^\nu(z) = \{(1-z/x_{-1}^\nu)(1-z/x_{+1}^\nu)\}^{1/2} \div (1-z/x_0^\nu)$$

can be defined to be real and positive along the upper sides of the β and negative on the lower sides. Let $\omega_{\gamma,A}(z)=u(z)+iAv(z)\prod\gamma^\nu(z)=u(z)+iAv_1(z)$, $A>0$. Let α_ν be the number of zeros of $u(z)$ in γ^ν. Then $\omega_{\gamma,A}(z)$ has $\sum\alpha_\nu$ zeros in $\mathfrak{G}_\beta$; it belongs to class B if and only if $\sum\alpha_\nu<\infty$, and to HB if and only if all $\alpha_\nu=0$.

The author then studies in detail the case when γ consists of the single interval $(-c, c)$. Then $\zeta=(z^2-c^2)^{1/2}$ maps $\mathfrak{G}_\beta$ on a two-sheeted Riemann surface $\mathfrak{R}$ over the upper half-plane, with branch points over $\zeta=ic$. Let $f(z)$ be admissible on $\mathfrak{R}$ and real on the real axis; let $\hat{\mathfrak{R}}$ be $\mathfrak{R}$ with its boundary. Let $|f[\zeta(x)]|<|\omega(x)|$ for $x\in\beta$ and $|f[\zeta(z)]/\omega(z)|<1$ as $z\to\infty$ in the upper half-plane. The author calculates the number of zeros of $u[z(\zeta)]-f(\zeta)$ on $\hat{\mathfrak{R}}$. In particular, if $|\omega(x)|=$ constant on the real axis and $f(0)=u(-c)$ on one sheet of $\hat{\mathfrak{R}}$ and $f(0)=u(c)$ on the other, then if $|\zeta|\leq c$ and $v(x_0)=0$, $-c<x_0<c$, $f(\zeta)\neq u(x_0)$. For the case when $\omega(z)=e^{i\sigma z}$ and $f(z)$ is an entire function of exponential type these theorems contain results of Hörmander's [Math. Scand. **3** (1955), 21–27; MR **17**, 247]. *R. P. Boas, Jr.* (Evanston, Ill.)

5010:

Meĭman, N. N. On the theory of analytic functions with least deviation from zero in a domain. Dokl. Akad. Nauk SSSR **126** (1959), 274–277. (Russian)

[See the preceding review and references given there.] An admissible $f(z)$ is called a minorant of $\omega(z)$ if $\sup|f(z)/\omega(z)|$ and $\sup|\bar{f}(z)/\omega(z)|<+\infty$. The deviation of $f(z)$ from 0 with weight $\omega(z)$ is $M(f;\omega)=\sup|f(z)/\omega(z)|$. Let D be a class of minorants of $\omega(z)$, closed under con-

vergence in $\mathfrak{G}$ or $\tilde{\mathfrak{G}}$. The least deviation from zero of functions of D is $L(\omega; D)=\inf M(f;\omega)$ when f runs through D. The author states the following general principle: the solutions of all problems on least deviation follow from properties of functions of HB or limits of such functions. Several examples are given, both new and old, including the following theorem. Let $f(z)$ be a real admissible function in $\mathfrak{G}$ or $\tilde{\mathfrak{G}}$ and $\omega(z)=u(z)+iv(z)\in \mathrm{HB}$. Then at every point $z_0\in\mathfrak{G}$ or $\tilde{\mathfrak{G}}$ at which $f(z_0)/u(z_0)$ is real we have $|f(z_0)/\omega(z_0)|\leqq M(f;\omega)$ and if there is equality, $f(z)\equiv \pm Mu(z)$. Further inequalities are obtained for derivatives, including the "Markov theorem" of Ahiezer and Levin [same Dokl. **117** (1957), 735–738; MR **20** #971] and the following: if $f(z)$ is a real entire function of order $\frac{1}{2}$, type σ, with $|f(x)|\leqq 1$ for $x\geqq 0$, then $|f'(x)|\leqq\frac{1}{2}\sigma^2$ for $x\geqq 0$ and equality occurs only for $\pm\cos\sigma x^{1/2}$.

R. P. Boas, Jr. (Evanston, Ill.)

5011:

Erdös, Paul; and Rényi, Alfréd. On singular radii of power series. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), 159–169. (Hungarian and Russian summaries)

Let $\mathcal{R}_u$ denote the class of functions (1) $f(z)=\sum c_k z^{n_k}$ that are regular and unbounded in the unit disk D. If $f\in\mathcal{R}_u$, let $e^{i\theta}$ be called a B-singular point of f provided, for each neighborhood U of $e^{i\theta}$, f is unbounded in $U\cap D$. Gaier and Meyer-König [Jber. Deutsch. Math. Verein. **59** (1956), Abt. 1, 36–48; MR **18**, 385] have shown that if $f\in\mathcal{R}_u$ and (1) has Hadamard gaps, then each point on the unit circle C is B-singular for f; but Erdös has constructed a function (1) in $\mathcal{R}_u$ for which $n_{k+1}-n_k\to\infty$ and for which the point $z=1$ is the only B-singular point on C [ibid. **60** (1957), Abt. 1, 89–92; MR **19**, 1045]. The authors now seek conditions on $\{n_k\}$ that ensure the existence of a function (1) in $\mathcal{R}_u$ for which the point $z=1$ is the only B-singular point on C.

There exist no sufficient conditions that involve only the rate at which $n_k/k\to 0$ as $n_k\to\infty$. For if (1) is in $\mathcal{R}_u$ and if, for each natural number m, all except finitely many of the exponents n_k are divisible by 2^m, then each point on C is B-singular. To eliminate number-theoretic complications, the authors consider the class of functions

$$(2)\qquad \sum c_k z^{n_k+\nu_k},$$

subject to appropriate restrictions on $\{\nu_k\}$. Theorem: If

$$(3)\qquad \liminf_{k-j\to\infty}(n_k-n_j)^{1/(k-j)} = 1,$$

then, for each sequence $\{\omega_k\}$ tending to ∞, there exists a function (2) in $\mathcal{R}_u$ such that $0\leqq\nu_k\leqq\omega_k$ and such that $z=1$ is the only B-singular point of f on C. The theorem becomes false if the condition $0\leqq\nu_k\leqq\omega_k$ is replaced by boundedness of $\{\nu_k\}$. The question whether the condition (3) is necessary for the existence of a function (2) with only one B-singular point on C remains open.

The proof of the theorem depends on the following lemma: Let $m_1<m_2<\cdots<m_d$ be natural numbers, and let $\nu_1,\nu_2,\cdots,\nu_d$ denote independent random variables, each of which takes the values $0, 1, \cdots, s-1$ with equal probability $1/s$. Let z be a number lying in $|z|\leqq 1$ and in $|1-z|\geqq 1/2s$, and let

$$Z=\sum_{j=1}^{d} z^{m_j+\nu_j}.$$

Then the probability that $|Z|\geqq 4d/s|1-z|$ is at most $4\exp(-d/32s^2)$.

G. Piranian (Ann Arbor, Mich.)

5012:

Iliev, Lyubomir. A test for non-continuable power series. Dokl. Akad. Nauk SSSR **126** (1959), 13–14. (Russian)

The test is based on the following lemma proved by a variation of the method of Szegö [Math. Ann. **87** (1922), 90–111]. If $f(z)=\sum a_n z^n$ is regular for $|z|<1$ and on part of $|z|=1$, and if $a_{n_\nu-k}=a_{m_\nu-k}$ $(k=1,2,\cdots[\theta n_\nu])$ for a sequence n_ν and associated $m_\nu>n_\nu+\theta n_\nu$, then

$$|a_{n_\nu}-a_{m_\nu}| < (1-\delta)^{n_\nu}$$

for some $\delta>0$ and sufficiently large ν.

A. J. Macintyre (Cincinnati, Ohio)

5013:

Lehner, Joseph. The Fourier coefficients of automorphic forms belonging to a class of horocyclic groups. Michigan Math. J. **4** (1957), 265–279.

Let Γ be a discrete subgroup of the group of all linear transformations $z\to Vz=(az+b)/(cz+d)$ with real coefficients a, b, c, d, where z lies in the upper half-plane $\Im(z)>0$. Suppose that Γ is not properly discontinuous at any point (including ∞) of the real axis. The transformations of Γ can be represented by unimodular matrices $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ with $c>0$, and $a>0$ if $c=0$. The (discrete) set of numbers c occurring in matrices of Γ is denoted by C. For a given $c\in C$ the set of all d such that there exists a matrix $V_{c,d}=\begin{pmatrix} \cdot & \cdot\\ c & d\end{pmatrix}\in\Gamma$ is denoted by D_c. The author makes the assumption that ∞ is a parabolic fixed point of Γ and that all parabolic fixed points of Γ are equivalent under Γ. Entire automorphic forms of real dimension r and multiplier system $\varepsilon(V)$ are analytic functions $F(z)$ regular in the upper half-plane and satisfying the equation $F(Vz)=\varepsilon(V)(-i(cz+d))^{-r}F(z)$ for every $V\in\Gamma$. It is supposed that the $\varepsilon(V)$ have absolute value 1 and are independent of z. If $Sz=z+\lambda$ $(\lambda>0)$ is the generating element of the cyclic subgroup of Γ consisting of all transformations of Γ which preserve ∞, then $t=\exp 2\pi iz/\lambda$ is a uniformizing variable at ∞. If $(-i)^{-r}\varepsilon(S)=\exp 2\pi i\alpha$, $0\leqq\alpha<1$, then $F(z)\cdot\exp(-2\pi i\alpha z/\lambda)$ has a Laurent expansion in t, which is supposed to contain only a finite number of terms with negative powers of t. Thus $F(z)=\sum_{m=-\mu}^{\infty}a_m\exp 2\pi i(m+\alpha)z/\lambda$. The author applies the Hardy-Littlewood circle method for the determination of the a_m with $m\geqq 0$ in terms of the a_m with $m<0$. By means of a suitable dissection (dependent on Γ) of a line segment $0\leqq x\leqq\lambda$, $y=y_0$ $(z=x+iy)$ instead of the usual Farey dissection he obtains the following results. (1) If $F(z)$ is an automorphic form of positive dimension r, then for $m\geqq 0$ we have

$$\lambda a_m = 2\pi\sum_{\nu=1}^{\mu}a_{-\nu}\sum_c c^{-1}A_{c,\nu}(m)L_c(m,\nu,r,\alpha),$$

where c runs through those $c\in C$ which are positive and where

$$A_{c,\nu}(m)=\sum_{d\in D_c}\varepsilon^{-1}(V_{c,d})\cdot\exp 2\pi i[(m+\alpha)d-(\nu-\alpha)a]/c\lambda,$$

$$L_c(m, \nu, r, \alpha) = \left(\frac{\nu-\alpha}{m+\alpha}\right)^{(r+1)/2} I_{r+1}(4\pi c^{-1}\lambda^{-1}(\nu-\alpha)^{\frac{1}{2}}(m+\alpha)^{\frac{1}{2}})$$
$$(m+\alpha>0,\ r>0),$$
$$L_c(0, \nu, r, 0) = \Gamma^{-1}(r+2)(2\pi\nu/c\lambda)^{r+1}$$

and I_r is the Bessel function with purely imaginary argument. This theorem was obtained by Petersson by different methods [S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1950**, 417–494; MR **12**, 806]. (2) If $F(z)$ is an automorphic form of dimension 0, then

$$\lambda a_m = 2\pi \sum_{\nu=1}^{\mu} a_{-\nu} \sum_c c^{-1} A_{c,\nu}(m) L_c(m, \nu, 0, \alpha) + O(1) \quad (m \geq 1),$$

where the summation is over those $c \in C$ for which $0<c<\beta\sqrt{m}$ and β is any positive constant. (3) If $F(z)$ is an automorphic form of dimension -2 and is the derivative of a form of dimension zero, then

$$a_m = (2\pi/\lambda)^2 i(m+\alpha) \sum_{\nu=1}^{\mu} a_{-\nu} \sum_c c^{-1} A_{c,\nu}(m) L_c(m, \nu, 0, \alpha)$$
$$(m \geq 1),$$

where the summation is again over those $c \in C$ for which $0<c<\beta\sqrt{m}$. The author considers also the case $r<-2$ (for negative r the definition of $L_c(m, \nu, r, \alpha)$ must be modified; cf. the correction in the second part of the paper [below]. *H. D. Kloosterman* (Leiden)

5014:

Lehner, Joseph. The Fourier coefficients of automorphic forms on horocyclic groups. II. Michigan Math. J. **6** (1959), 173–193.

The author extends the results in the first part of the paper [reviewed above] to the case in which the group Γ has a finite number of inequivalent parabolic fixed points. *H. D. Kloosterman* (Leiden)

5015:

Montel, Paul. Sur les éléments exceptionnels des fonctions analytiques. C. R. Acad. Sci. Paris **249** (1959), 7–8.

Let $u=f(z)$ be regular in the open plane. The author considers $f(z)$ as defining a curve Γ in the projective plane of two complex variables, and tangent and normal to Γ at the point $(z, f(z))$ are defined, respectively, as

$$U-f(z) = f'(z)(Z-z), \quad (Z-z)+f'(z)(U-f(z)) = 0.$$

Let P_1, P_2, P_3 be distinct collinear points in the (U, Z) plane. The author states a number of theorems of which the following are typical. (1) If Γ is such that its tangent never passes through any of P_1, P_2, P_3, then $f(z)$ is linear. (2) For any curve Γ a normal always passes through at least one of P_1, P_2, P_3. (3) If the chords through the points over z, $z+h$ for fixed h never pass through any of P_1, P_2, P_3, then $f(z)=\omega(z)(z-\alpha)+\beta$, where $\omega(z)$ is periodic of period h. Analogous results for curves of contact of higher order and applications to the theory of normal families are also given, but no proofs. *W. K. Hayman* (London)

5016:

Kahane, Jean-Pierre; et Rubel, Lee. Sur les produits canoniques de type nul sur l'axe réel. C. R. Acad. Sci. Paris **248** (1959), 3102–3103.

Let $C(z)$ be the even entire function $\prod(1-z^2/\lambda_n^2)$ with real zeros $\pm\lambda_1$, $\pm\lambda_2$, $\cdots$. It is assumed that $D(t)=n(t)/t$ is bounded. Put

$$h(C, \theta) = \limsup_r r^{-1} \log |C(re^{i\theta})|,$$

and $h(C)=h(C, \pi/2)$; C is of exponential type $h(C)$. It is well known [Boas, *Entire functions*, Academic Press, New York, 1954; MR **16**, 914; Chap. 8] that the existence of density $D=\lim_t D(t)$ for the zeros is a very strong condition, being in fact equivalent to the convergence of $\int^\infty r^{-2}\log^+|C(r)|dr$ and of $\lim_r r^{-1}\log|C(re^{i\theta})|$. The authors have shown that for any increasing function $T(r)$ such that $T(r)/r$ and $(\log r)/T(r)$ increase and $\int r^{-2}T(r)dr$ diverges, one can construct canonical products $C(z)$ with $C(r)=O(\exp T(r))$ and such that

$$\limsup_{t\to\infty} D(t) = h(C)/\pi > 0, \quad \liminf_{t\to\infty} D(t) = 0$$

and

$$\liminf_{r\to\infty} r^{-1}\log|C(re^{i\theta})| = 0$$

for all $\theta\neq 0$. This is then applied to show that there can exist functions C_1, C_2 so that $h(C_1C_2)=h(C_1)=h(C_2)$. *R. C. Buck* (Princeton, N.J.)

5017:

Koosis, Paul. Nouvelle démonstration d'un théorème de Levinson concernant la distribution des zéros d'une fonction de type exponentiel, bornée sur l'axe réel. Bull. Soc. Math. France **86** (1958), 27–40.

If an entire function $F(z)(z=x+iy)$ is bounded on the real axis and is of exponential type A both for $y>0$ and $y<0$, and if $n_+(r)$ and $n_-(r)$ denote the number of its zeros of modulus $<r$ for $x>0$ and $x<0$, respectively, then $n_+(r)/r\to A/\pi$ and $n_-(r)/r\to A/\pi$ as $r\to\infty$. For this result, due to N. Levinson, a comparatively simple proof is given, based only on Jensen's Theorem, on properties of the Blaschke product $B(z)$ and on the Poisson integral. It is shown that, with some measure $\mu(t)$, for $y>0$,

$$\log|F(x)| = Ay + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\,d\mu(t)}{(x-t)^2+y^2} + \log|B(z)|,$$

that

$$R^{-1}\int_0^\pi \log|B(Re^{i\varphi})|d\varphi \to 0 \quad (R\to\infty)$$

and, by means of Jensen's Theorem, the Poisson integral and a suitably constructed sequence R_n of numbers, that

$$R^{-1}\int_0^R \frac{n(r)}{r}dr \to \frac{2A}{\pi}\ (R\to\infty;\ n(r) = n_+(r)+n_-(r));$$

whence, by an argument due to Landau, $n(R)/R\to 2A/\pi$. Let $\{z_n=r_ne^{i\theta_n}\}$ be the sequence of the zeros of $F(z)$. By the convergence of $S(r)=\sum_{r_n<r} r_n^{-1}\cos\theta_n$ to a limit as $r\to\infty$, it is shown that $n_+(R)-n_-(R)=o(R)$, which completes the proof. Finally the result is stated in a detailed and generalised form. *H. Kober* (Birmingham)

5018:

Roux, Delfina. Sul minimo modulo delle funzioni intere di genere zero. Riv. Mat. Univ. Parma **8** (1957), 227–250. (English summary)

Let $f(z)$ be an entire function of genus 0 with $f(0)=1$. Let, as usual, $m(r)$ be the minimum modulus, $n(r)$ the

number of zeros, $N(r)=\int_0^r t^{-1}n(t)dt$, and put $Q(r)=r\int_r^\infty t^{-2}n(t)dt$. It is known that $\log m(r)\geqq N(r)-\Delta(r)Q(r)$ on a set of unit linear density, where $\Delta(r)\to\infty$ arbitrarily slowly; the author constructs an example showing that $\Delta(r)$ cannot be replaced by a constant. Her main result is that $\log m(r)\geqq N(r)-KQ(r)$ on a set of unit linear density provided that the moduli r_k of the zeros are sufficiently uniformly distributed in the following sense: there are constants C_1 and C_2 and a function H such that $H(R)\to\infty$, $\{1/H(R)\}\log\{R/H(R)\}\to 0$, $n''(R)/n(R)\to 0$ and $\psi(H;C_1,C_2)\to\infty$, where n'' and ψ are defined as follows. Divide $(0, R)$ into $H(R)$ equal intervals, and let classes I and II consist of those intervals that contain respectively less and more than $C_1n(R)/H(R)$ points r_k. Let ν be the number of intervals (II) and n'' the number of r_k belonging to them. Then

$$\psi(H;C_1,C_2) = C_2/\{(n''/n)-(C_1\nu/H)\}-\log\nu.$$

R. P. Boas, Jr. (Evanston, Ill.)

5019:

Arakelyan, N. U. Refinement of some Keldyš theorems on asymptotic approximation by entire functions. Dokl. Akad. Nauk SSSR **125** (1959), 695–698. (Russian)

The author generalizes some theorems of Keldyš [same Dokl. **47** (1945), 239–241; MR **7**, 150; see also Mergelyan, Uspehi Mat. Nauk. (N.S.) **7** (1952), no. 2(48), 31–122; MR **14**, 547] along lines suggested by Džrbašyan [Dokl. Akad. Nauk SSSR **111** (1956), 749–752; MR **19**, 138]. A continuum E is said to satisfy condition B if there exists $r(t)>0$, $r(t)\to\infty$, such that every point z of the complement of E can be joined to ∞ by a Jordan arc in E and outside a disk of center 0 and radius $r(|z|)$. (1) Let E satisfy condition B, let f be continuous on the finite part of E and analytic in its interior; let $P(r)$ be nondecreasing with $\int_1^\infty r^{-3/2}P(r)dr<\infty$. Then for every $\varepsilon>0$ there is an entire $G(z)$ for which $|f(z)-G(z)|<\varepsilon e^{-P(|z|)}$. The theorem fails if the integral diverges. (2) If E lies in $|\arg z|\leqq\alpha/2$, the integral in (1) is to be replaced by $\int_1^\infty r^{-1-\pi/\alpha}P(r)dr$. (3) If the angle is replaced by the strip $\Im|(z)|\leqq h/2$ the integral is to be replaced by $\int_0^\infty P(x)e^{-\pi x/h}dx$. (4) If E lies in the region D bounded by the spirals $r=r_1e^{m\theta}$, $r=r_2e^{m\theta}$, where $m>0$, $0<r_1<r_2<r_1e^{2\pi m}$, the integral is to be replaced by $\int_1^\infty P(r)r^{-\rho}dr$, where $\rho=1+\pi(m+m^{-1})/\log(r_2/r_1)$.

R. P. Boas, Jr. (Evanston, Ill.)

5020:

Banaschewski, Bernhard. Zur Idealtheorie der ganzen Funktionen. Math. Nachr. **19** (1958), 136–160.

Let $\mathfrak{S}$ denote an ideal in the lattice of all subsets of a nonempty set E, and let W denote an abelian (additive) cancellation semigroup with an identity element 0 that is totally ordered by letting $a\leqq b$ if and only if there is a $c\in W$ such that $a+c=b$. Let $\mathfrak{b}=\mathfrak{b}(E,\mathfrak{S},W)$ denote the set of all functions f defined on E with values in W such that $\{x\in E: f(x)\neq 0\}\in\mathfrak{S}$. With operations defined coordinate-wise, $\mathfrak{b}$ becomes a lattice-ordered cancellation semigroup whose identity element (the constant function 0) is its smallest element. A subset $\mathfrak{a}$ of $\mathfrak{b}$ such that $f\wedge g\in\mathfrak{a}$ whenever $f,g\in\mathfrak{a}$, and $h\in\mathfrak{a}$ whenever $h\geqq f\in\mathfrak{a}$, is called an ideal of $\mathfrak{b}$. An ideal of $\mathfrak{b}$ is called primary if $f+g\in\mathfrak{a}$ and $f\notin\mathfrak{a}$ imply that there is a positive integer m such that $mg\in\mathfrak{a}$; in case m may be taken to be 1, $\mathfrak{a}$ is called a prime ideal. If $\mathfrak{a}$ is an ideal of $\mathfrak{b}$, and k is a positive integer, then $\mathfrak{a}^k$ denotes the set of all sums of k elements of $\mathfrak{a}$.

The author is concerned, for the most part, with the prime, primary, and maximal ideals of $\mathfrak{b}$, and makes a thorough study of them, with particular attention paid to the case when $W=N$ is the additive semigroup of nonnegative integers. A few sample results follow.

Every primary ideal of $\mathfrak{b}$ is contained in exactly one maximal ideal. The set of all primary ideals contained in a given maximal ideal $\mathfrak{m}$ is totally ordered under set inclusion. This totally ordered set is investigated thoroughly. Every intersection of prime [resp. primary] ideals contained in $\mathfrak{m}$ is prime [resp. primary]. In case $W=N$, every primary ideal that is not prime takes the form $\mathfrak{m}^k$ for some positive integer $k\geqq 2$, $\mathfrak{p}^2=\mathfrak{p}$ for every nonmaximal prime ideal $\mathfrak{p}$, and the intersection of all the powers of $\mathfrak{p}$ is either empty or is the largest prime ideal properly contained in $\mathfrak{p}$.

Let Γ denote the ring of entire functions, and let E denote the complex plane. If $0\neq\phi\in\Gamma$ and $x\in E$, let $f(x)=0$ if $\phi(x)\neq 0$, and let $f(x)$ denote the multiplicity of x as a zero of ϕ, if $\phi(x)=0$. The mapping $\phi\to f$ is a homomorphism of the multiplicative semigroup of nonzero elements of Γ onto $\mathfrak{b}(E,\mathfrak{S},N)$, where $\mathfrak{S}$ denotes the family of all closed, discrete subsets of E. Moreover, this mapping induces a one-one correspondence between the ideals of the ring Γ and those of the lattice-ordered semigroup $\mathfrak{b}(E,\mathfrak{S},N)$. Thus, the author's results subsume and generalize most of the reviewer's results on the prime ideals of Γ [Pacific J. Math. **3** (1953), 711–720; MR **15**, 537].

M. Henriksen (Lafayette, Ind.)

5021:

★**Lohwater, A. J. The cluster sets of meromorphic functions.** Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 171–177. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

This is a survey of the theory of cluster sets. The author treats mainly the case of meromorphic functions in the unit circle. Collingwood's result on prime ends [J. London Math. Soc. **31** (1956), 344–349; MR **18**, 201], Bagemihl's result on ambiguous points of arbitrary functions [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 379–382; MR **16**, 1095] and the reviewer's results on a new boundary cluster set [ibid., 398–401; MR **17**, 143] are cited. A new theorem of the author is announced and some problems are raised.

K. Noshiro (Nagoya)

5022:

Piranian, G. On a problem of Lohwater. Proc. Amer. Math. Soc. **10** (1959), 415–416.

Let $f(z)$ be a complex-valued function defined in $|z|<1$, let $\gamma_1(\vartheta),\cdots,\gamma_n(\vartheta)$ be n arcs lying in $|z|<1$ and terminating at the point $e^{i\theta}$, and let $C_{\gamma_i}(f,e^{i\theta})$, $i=1,\cdots,n$, denote the curvilinear cluster sets of $f(z)$ at $e^{i\theta}$ along $\gamma_i(\vartheta)$. The reviewer [#5021 above] indicated that it would be of interest to determine whether an analytic function has the property that, for a fixed $n>2$, the intersection $\bigcap_{i=1}^n C_{\gamma_i}(f,e^{i\theta})$ can be empty for a non-denumerable set of $e^{i\theta}$ on $|z|=1$. The author constructs a continuous function in $|z|<1$ which gives a negative answer to the problem—a discontinuous example had been given earlier by V. Jarník [Fund. Math. **27** (1936), 147–150; see p. 149]

—and announces that an analytic counterexample is forthcoming. *A. J. Lohwater* (Houston, Tex.)

5023:

Zamorski, J. Equations satisfied by the extremal star-like functions. Ann. Polon. Math. **5** (1958/59), 285–291.

The class of univalent functions $F(z)$ of form $z^{-1}+b_0+b_1z+\cdots$ for $0<|z|<1$, mapping the disc $|z|<1$ on a region starlike with respect to infinity, is denoted by G. The subclass of functions of form $z^{-1}(1-\sigma_1z)^{\beta_1}\cdots(1-\sigma_mz)^{\beta_m}$, mapping $|z|<1$ on the plane minus m finite rays from the origin, is denoted by H_m. The author writes $b_k=x_k+iy_k$ and considers real functions

$$E(x_1, y_1, \cdots, x_n, y_n),$$

which can be considered as functionals on G. Necessary conditions are obtained that E, when restricted to H_m, have an extremum for a particular function of H_m. From the form of these conditions it is deduced that $m\leqq n+1$ and furthermore that E takes on its extreme values in G at functions in H_{n+1}. Application of these results to the case $E=\operatorname{Re} b_n$ leads to relatively simple conditions for the extremal functions. These are solved for $n=1$, and it is verified that $|b_1|<1$ for the extremal function.

W. Kaplan (Ann Arbor, Mich.)

5024:

Newman, D. J. Some remarks on the maximal ideal structure of H^∞. Ann. of Math. (2) **70** (1959), 438–445.

H^∞ is the Banach algebra of all bounded analytic functions in the open unit disc D. The "corona" is defined as the complement of the closure of D in the maximal ideal space of H^∞. It is not known whether the corona is empty or not. The author contributes the following theorem to this problem: If M is a maximal ideal in H^∞ which contains no Blaschke product, then M is not in the corona. (Hypothesis and conclusion are unfortunately interchanged in the statement of theorem 3.) He also shows that the maximal ideals in H^∞ which contain no Blaschke products are precisely those which can be extended to maximal ideals in L^∞, the algebra of all bounded measurable functions on the unit circle.

W. Rudin (Madison, Wis.)

5025:

Šaginyan, A. L. A fundamental inequality in the theory of functions and its applications. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz-Mat. Nauk **12** (1959), no. 1, 3–25. (Russian. Armenian summary)

Let $f(z)$ be regular and of bounded characteristic in $|z|<1$, and satisfy for the sake of simplicity $f(0)=1$. Let L be any system of arcs meeting no circle $|z|=\rho$ for $0<\rho<1$ more than once. Let $\omega(t)$, $0<t<1$, be continuous, positive and decreasing, and satisfy the conditions $\omega(0)=1$ and

$$(1) \qquad \int_e \frac{\omega(t)dt}{1-t} < \infty,$$

where e is the set of t for which the circle $|z|=t$ meets L. Then the fundamental inequality established by the author can be written in the form

$$(2) \quad \int_L \omega(|z|)\log|f(z)|d|z| \geq \frac{1}{4\pi}T(1,f)\int_e (1-t)\omega(t)dt$$

$$-\frac{1}{\pi}\left(\int_e \frac{\omega(t)dt}{1-t}\right)m\left(1,\frac{1}{f}\right)-\left(A_1+A_2\int\frac{\omega(t)dt}{1-t}\right)(T(1,f)+1),$$

where A_1, A_2 are absolute constants. The result is obtained by applying the Poisson-Jensen formula to $f(z)$, multiplying both sides by $\omega(|z|)$ and integrating along L. The first two terms on the right-hand side of (2) arise from the contribution of the boundary integral, the last term from the contribution of the zeros of $f(z)$. This term is somewhat more troublesome to estimate.

The author makes a variety of applications of his basic result. He notes first of all that under the conditions given the left-hand side of (2) is always finite. Taking L to be a set on the real axis and $f(z)=\exp[-1/(1-z)]$ he shows that the condition (1) is essential for this. As a further example, if $\omega(t)$ decreases in [0, 1] and $\int_0^1 \frac{\omega(t)dt}{1-t}<+\infty$, the author constructs a Blaschke product $B(z)$ such that

$$\int_0^1 \log|B(x)|\omega(x)dx = -\infty.$$

The author next obtains an inequality of the Milloux type for functions of his class in terms of $T(1,f)$ and the left-hand side of (2) using the formula (2). This application has however the disadvantage that (2) depends in general in a rather complicated way on the behaviour of $f(z)$ at the origin. Another application (theorem 2) is an immediate consequence of Heins's theorem that the minimum modulus $\mu(r)$ of a bounded function f cannot satisfy

$$(1-r)\log\mu(r)\to-\infty, \quad \text{as} \quad r\to 1,$$

except when $f(z)=0$ [Duke Math. J. **14** (1947), 179–215; MR **8**, 575].

Another application is obtained by taking for L a variable radius $z=re^{i\phi}$, integrating with respect to ϕ and so obtaining a double integral on the left-hand side. It seems to the reviewer that other important applications may well be made in the future. The inequality (2) no doubt extends to subharmonic and meromorphic functions.

The paper is marred by a large number of minor arithmetical and other inaccuracies so that the basic lemma on page 10 is proved only with somewhat larger constants than are contained in the statement. Also the class of functions defined by the author at the beginning of his paper is somewhat narrower than those of bounded characteristic, since the Poisson-Jensen formula is applied on $|z|=1$, but the author then uses $f(z)=\exp[-1/(1-z)]$, which does not belong to the narrower class, as a counter example to his theorems. However, since the author's result extends to the class of functions of bounded characteristic without difficulty, this and the other points are not of great importance. In essentials the paper is clear and easy to read. *W. K. Hayman* (London)

5026:

Havin, V. P. On a problem of V. V. Golubev. Dokl. Akad. Nauk SSSR **126** (1959), 511–513. (Russian)

Let F be a bounded, closed set of points of the extended complex plane R and $\mu(e)$ a nonnegative measure defined on subsets of F and such that $\overline{F\backslash e}=F$ whenever e is a subset of measure 0. The author proves that the class of functions, regular in $R\backslash F$ and vanishing at $z=\infty$, coincides with the class of functions representable for z

not in F by series of the form $\sum_{p=0}^{\infty} \int_F y_p(t)d\mu_t/(t-z)^{p+1}$, where

$$\lim_{p\to\infty} \|y_p\|^{1/p} = 0,$$

$$\|y_p\|^2 = \int_F |y_p(t)|^2 d\mu < \infty, \quad p = 0, 1, 2, \cdots.$$

This result includes a verification of a conjecture of V. V. Golubev as a special case. When L is a closed Jordan curve and $u(z)$ is regular in its interior, then (putting $u(z)\equiv 0$ exterior to L) the above result implies the representation (for z not on L)

$$u(z) = \sum_{p=0}^{\infty} \int_L \frac{y_p(\zeta)|d\zeta|}{(\zeta-z)^{p+1}}, \lim_{p\to\infty} \left(\int_L |y_p(\zeta)|^2|d\zeta|\right)^{1/2p} = 0.$$

In this connection the author recalls the result of B. A. Vostrecov [Dokl. Akad. Nauk SSSR **65** (1949), 7–8; MR **10**, 523] when L is the unit circle, namely the representation

$$u(z) = \int_0^{2\pi} \varphi(e^{i\theta})g(1/(e^{i\theta}-z))de^{i\theta},$$

where g is an integral function and $\varphi(e^{i\theta}) \in L^2(0, 2\pi)$.

J. F. Heyda (Cincinnati, Ohio)

5027:

★**Nassif, M. Products and zeros of basic sets.** University of Assiut Monograph Series, No. 2. Assiut, 1958. xii+142 pp.

This is an appropriate continuation to the earlier monographs by Whittaker [*Interpolatory function theory*, Cambridge Univ. Press, London, 1935; *Sur les séries de base de polynomes quelconques*, Gauthier-Villars, Paris, 1949; MR **11**, 344]. With a polynomial set $\{p_n\}$, itself a basis for the linear space of polynomials, one associates a matrix (π_{nk}) and linear functionals Π_k, where $z^n = \sum_k \pi_{nk}p_k(z)$ and

$$(*) \qquad \Pi_k(f) = \sum_n \frac{f^{(n)}(0)}{n!} \pi_{nk}.$$

The matrix $\Pi = (\pi_{nk})$ has inverse $P = (p_{kn})$, where $p_k(z) = \sum p_{kn}z^n$. With any analytic functions f, regular at the origin, one associates the so-called basic series $\sum_k \Pi_k(f)p_k$. A set $\{p_n\}$ is said to be effective in the closed disc $D: |z| \leq R$ if every function f which is regular on D is represented there by the basic series; this requires that the series $(*)$ defining the coefficient functionals $\Pi_k(f)$ must converge, and that the basic series itself must converge to f on D. The central theme of the early chapters of the present monograph is the relationship between the effectiveness of polynomial sets with coefficient matrices P_1 and P_2, and that with matrix P_1P_2. Most of the results detailed here have been previously published by the author and his compatriots. The theorems are amply illustrated by many examples. Chapter IV treats the related problem of describing the Whittaker order of growth of the polynomial set with matrix P_1P_2, in terms of that of P_1 and P_2. In Chapter V, an analogous approach is made to study the effectiveness of specific polynomial sets in a non-circular region D by means of the Faber polynomials associated with D. The remaining two chapters deal with the dependency of effectiveness and order upon the location of the zeros of the polynomials $\{p_n\}$. This in turn is related to the problems of determining the exact value of the Whittaker constant W and the Gontcharov constant G; the conjecture $W = 1/G = 2/e$ is still open.

R. C. Buck (Princeton, N.J.)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

5028:

Look, K. H. Schwarz lemma and analytic invariants. Sci. Sinica **7** (1958), 453–504.

This paper provides proofs for theorems reported in previous notes of the author. Let D be a bounded schlicht domain in the space C^n of n complex variables, and let $T(z, \bar{z})$ be the metric matrix of D, so that $ds^2(z, \bar{z}) = dzT(z, \bar{z})\overline{dz}$ is the Bergmann metric of D. Sections 1–3 [Sci. Record (N.S.) **1** (1957), no. 2, 5–8; MR **19**, 644] are based on the following theorem: Let $w=f(z)$ be an analytic mapping of D into C^n such that

$$\sum |f_i(z)|^2 \leq M^2;$$

then

$$(\partial f/\partial z)\overline{(\partial f'/\partial z)} \leq M^2T(z, \bar{z}).$$

Sections 4–8 compute explicitly the Schwarz constant [defined in the review cited] for spheres, polycylinders, and the classical (homogeneous) domains. Sections 9–10 obtain certain analytic invariants for D which provide necessary and sufficient conditions that two irreducible classical domains be equivalent, and show the inequivalence of a large proportion of the list of such domains.

F. D. Quigley (New Orleans, La.)

SPECIAL FUNCTIONS

See also 5111, 5122.

5029:

★**Szegö, Gabor. Orthogonal polynomials.** American Mathematical Society Colloquium Publications, Vol. 23. Revised ed. American Mathematical Society, Providence, R.I., 1959. ix+421 pp.

Da ein vortreffliches Referat über die erste Auflage dieses Buches [Amer. Math. Soc., New York, 1939] bereits existiert [MR **1**, 14] so können wir uns kurz fassen. In der Zwischenzeit (20 Jahre!) ist kein Buch erschienen, das die orthogonalen Polynome zum ausschliesslichen Gegenstand hat, sondern nur allgemeinere Bücher, wie *Higher transcendental functions* [Bateman Manuscript Project, McGraw-Hill, New York, 1953–55; MR **15**, 419; **16**, 586; und Tricomi, *Vorlesungen über Orthogonalreihen* [Springer, Berlin-Göttingen-Heidelberg, 1955; MR **17**, 30], die mehrere Kapitel über die orthogonalen Polynome enthalten. Das vorliegende Buch bleibt also daher, sowie infolge seiner Qualitäten, das Werk über orthogonale Polynome.

Es ist zu bedauern, dass das Buch nicht noch mehr erweitert worden ist, entsprechend den inzwischen erzielten Fortschritten. In einem Appendix wird die einzige beträchtliche Hinzufügung zur ersten Auflage, nämlich die (wegen ihres "irregulären" oder "singulären" Verhaltens) wichtigen Polynome von Pollaczek, behandelt.

Ferner werden 22 neue Aufgaben und Übungen, die neues Material bringen, hinzugefügt. Die Bibliographie ist in selektiver Weise durch Hinzufügung eines neuen Literaturverzeichnisses erweitert worden. Schliesslich wurden zahlreiche kleinere Hinzufügungen und Verbesserungen gemacht.

Der Autor weist in der "Vorrede zur zweiten Auflage" auf die grossen Verdienste hin, die sich der inzwischen heimgegangene Professor J. D. Tamarkin um das vorliegende Buch, sowie allgemeiner um die amerikanische Mathematik, erworben hat. *B. Germansky* (Berlin)

5030:

Al-Salam, Waleed A. A generalization of some polynomials related to the theta functions. Riv. Mat. Univ. Parma 8 (1957), 381–395.

The paper is concerned with the polynomial $H_n^{(\nu)}(x)$ which satisfies

$$H_{n+1}^{(\nu)}(x) = (1+x)H_n^{(\nu)}(x) - (1-q^{n+\nu})xH_{n-1}^{(\nu)}(x),$$
$$H_0^{(\nu)}(x) = 1, \quad H_1^{(\nu)}(x) = 1+x$$

and the second polynomial $G_n^{(\nu)}(x)$ obtained from $H_n^{(\nu)}(x)$ by replacing q by q^{-1}. The special case $\nu=0$ was discussed by Szegö [S.-B. Preuss. Akad. Wiss., Phys.-Math. Kl. **1926**, 242–252] and Wigert [Ark. Math. Fys. **27** (1923), no. 18], and more recently by the reviewer [Ann. Mat. Pura Appl. (4) **41** (1956), 359–373; MR **17**, 1205]. The corresponding generalization of the Hermite polynomial was studied by Palamà [Riv. Mat. Univ. Parma **4** (1953), 363–386; MR **16**, 470] and Toscano [ibid. **6** (1955), 117–140; MR **17**, 1082].

The main results of the present paper are the following.

(1) $H_n^{(\nu)}(x)$ is a polynomial in the two variables x and $z=q^\nu$:

$$H_n^{(\nu)}(x) = \sum_s \sum_k q^{k^2} \begin{bmatrix} s \\ k \end{bmatrix} \begin{bmatrix} n-s \\ k \end{bmatrix} x^s z^k,$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(1-q^n) \cdots (1-q^{n-r+1})}{(1-q) \cdots (1-q^r)}.$$

(2) For $m \leq n$,

$$H_m(x)H_n^{(\nu)}(x) = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n+\nu \\ r \end{bmatrix} (q)_r x^r H_{m+n-2r}^{(\nu)}(x)$$

where $(a)_n = (1-a)(1-qa) \cdots (1-q^{n-1}a)$; also the inverse formula.

(3) An explicit formula for

$$(H_n^{(\nu)}(x))^2 - H_{n+1}^{(\nu)}(x)H_{n-1}^{(\nu)}(x).$$

(4) A characterization of $H_n^{(\nu)}(x)$ as the unique polynomial solution of a certain functional equation.

(5) The mixed recursion formula:

$$H_{n+1}^{(\nu-1)}(x) = (1+x)H_n^{(\nu)}(x) - (1-q^\nu)xH_{n-1}^{(\nu+1)}(x).$$

(6) A q-analog of a formula of Toscano:

$$H_{n+1}^{(\nu)}(x) = \sum_{2r\leq n} (-1)^r x^r q^{\frac{1}{2}r(r+1)}(q^\nu)_r \begin{bmatrix} n-r \\ r \end{bmatrix} H_{n-2r}(x).$$

(7) If $\nu=-r$, a negative integer, then $H_n^{(-r)}(x) = G_r(x)H_{n-r}(x)$. Also $H_n^{(-r-\mu)}(x) = G_n^{(\mu)}(x)$ for arbitrary μ. For each of these results there is a companion result obtained on replacing q by q^{-1}.

In the above, $H_n(x) = H_n^{(0)}(x)$ and $G_n(x) = G_n^{(0)}(x)$.

L. Carlitz (Durham, N.C.)

5031:

Al-Salam, W. A.; and Carlitz, L. The expansion of certain products containing Bessel functions. Matematiche. Catania **12** (1957), 31–34.

This paper gives an expression for the coefficient in the general term of the Laurent expansion of

$$(-1)^n(2t/\pi)^{1/2}(2/t)^\mu K_{n+\frac{1}{2}}(t)I_\mu[t\{(x^2-1)(y^2-1)\}^{1/2}]e^{-txy}.$$

The coefficient can be expressed as a series of pairs of Legendre functions, and it generalizes a result due to Bailey. The rather old-fashioned style of the notation and printing make an elegant result appear complicated.

L. J. Slater (Cambridge, England)

5032:

Meulenbeld, B. Quadratic transformations of generalized Legendre's associated functions for special values of the parameters. Arch. Rational Mech. Anal. **3**, 460–471 (1959).

The well-known quadratic transformations of hypergeometric functions are applied to the functions indicated in the title. *A. Erdélyi* (Pasadena, Calif.)

5033:

Mihlin, S. G. Differentiation of series in spherical functions. Dokl. Akad. Nauk SSSR **126** (1959), 278–279. (Russian)

Let S denote the unit sphere in E_m, and let $f(x)$ be defined and homogeneous of degree zero in a spherical shell Ω containing S. Theorem 1: If $f(x) \in W_2^{(l)}(\Omega)$ [for terminology see S. L. Sobolev, *Nekotorye primeneniya funkcional'nogo analiza v matematičeskoi fizike*, Izdat. Leningrad. Gos. Univ., 1950; MR **14**, 565] then the expansion of $f(x)$ in spherical functions $Y_{n,m}^{(k)}$ can be differentiated l times with respect to the cartesian coordinates of the point x, and the differentiated series converges in the $L_2(\Omega)$-metric. Theorem 2: If $f(x) \in W_2^{(l)}(\Omega)$, $l \geq m-1$, then the expansion of $f(x)$ in spherical functions as well as the series obtained from it by differentiation of order $\leq l-m+1$ with respect to the cartesian coordinates of the point x converge uniformly and absolutely in Ω. *P. Henrici* (Los Angeles, Calif.)

5034:

Popov, Blagoj S. On ultraspherical polynomials. Boll. Un. Mat. Ital. (3) **14** (1959), 105–108.

The author gives an explicit formula for the product of two ultraspherical polynomials $P_n^{(\alpha,\alpha)}(x)P_m^{(\beta,\beta)}(x)$ as an expansion of ${}_4F_3$ and $P_{m+n-2k}^{(\alpha,\alpha)}(x)$ functions. The details of the method are not given, though presumably it is based on one of the standard interchanges in the order of summation. From this general result, the author deduces Bailey's expansion of the product of two associated Legendre functions $(1-x^2)^{-r/2}P_n^r(x)P_m^r(x)$, and a more general expansion of $(1-x^2)^{-r/2}P_m^r(x)P_n^s(x)$. He also gives an expansion of the integral

$$\int_{-1}^{+1} P_n^{(\alpha,\alpha)}(x)P_m^{(\beta,\beta)}(x)dx.$$

L. J. Slater (Cambridge, England)

5035:

Campbell, Robert. Une propriété des développements

en séries de polynomes hypergéométriques confluents. C. R. Acad. Sci. Paris **248** (1959), 3104–3105.

The author considers the Darboux-Christoffel formula for the partial sum $\sum_0^N a_nP_n(x)$ of the development of any function $f(x)$ in the space of polynomials $P_n(x)$. By a very interesting deductive method he proves that if there exists a certain relation between $P_n(x)$ and its first h derivatives then the $P_n(x)$ must in fact be confluent hypergeometric functions.

L. J. Slater (Cambridge, England)

ORDINARY DIFFERENTIAL EQUATIONS

See also 4889, 5153, 5299.

5036:

★Kamke, E. Differentialgleichungen. Lösungsmethoden und Lösungen. Teil I: Gewöhnliche Differentialgleichungen. 6. Aufl.; Teil II: Partielle Differentialgleichungen erster Ordnung für eine gesuchte Funktion. 4. Aufl. Mathematik und ihre Anwendungen in Physik und Technik. Reihe A. Bd. 18. Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1959. I. xxvi + 666 pp., DM 36.80; II. xv + 243 pp., DM 16.00.

Die 2. Auflage des I. Teiles, Leipzig, 1943, und die 3. Auflage des selben Teiles wurden besprochen in MR **9**, 587, 33. Die Veränderungen der 6. Auflage beziehen sich fast nur auf Einzelheiten, besonders in Teil C. Für Teil II siehe Besprechung der 1. Auflage, Leipzig, 1944 [MR **10**, 378]. Die vorliegende 4. Auflage ist unverändert.

5037:

Funato, Mitsuharu. On Duffings's equation. Math. Japon. **5** (1958/59), 29–34.

Consider Duffing's equation $x''+Ax+2Bx^3=E$, $x(0)=0$, $x'(0)=v_0>0$, where $A>0$, $B>0$ and $E>0$. Let X, $-\bar{X}$, T be the maximum, minimum, and period of the solution x, and $t_{\dot x}$, $-\bar{t}_{\dot x}$ the first positive zero and the first negative zero of $\dot{x}$. The author uses elementary methods to show that: (1) For fixed A, B and E, X and $\bar{X}$ increase and T, $t_{\dot x}$ and $\bar{t}_{\dot x}$ decrease as v_0 increases for $v_0>V_0$, where V_0 is a constant. (2) For fixed v_0, A and B, X increases and $\bar{X}$, T, $t_{\dot x}$ and $\bar{t}_{\dot x}$ decrease as E increases. (3) As A or B increases (all others fixed) X, $\bar{X}$, $t_{\dot x}$, $\bar{t}_{\dot x}$ and T all decrease. Certain refinements and extensions of these results are also given.

C. G. Maple (Ames, Iowa)

5038:

Brauer, Fred; and Sternberg, Shlomo. Errata to our paper "Local uniqueness, etc." Amer. J. Math. **81** (1959), 797.

The authors point out various errors in their previous paper [same **80** (1958), 421–430; MR **20** #1806] and correct some of them.

E. A. Coddington (Los Angeles, Calif.)

5039:

Tung, Chin-chu. Positions of limit-cycles of the system (E_2)

$$dx/dt=\sum_{0\le i+k\le 2} a_{ik}x^iy^k,\quad dy/dt=\sum_{0\le i+k\le 2} b_{ik}x^iy^k.$$

Sci. Record (N.S.) **2** (1958), 421–425.

Several theorems are stated concerning limit cycles of the system $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$, where P and Q are polynomials of second degree; some of the proofs are outlined and a full exposition is said to be in preparation. The principal theorem is that, for proper choice of P, Q, there are three limit cycles $C_i (i=1,2,3)$ whose interiors I_i satisfy the relations $I_1\cap I_2\neq 0$, $I_1\cap I_3=0$. The paper is related to one of Petrovskiĭ and Landis [Dokl. Akad. Nauk SSSR **102** (1955), 29–32; MR **16**, 1110].

W. Kaplan (Ann Arbor, Mich.)

5040:

Tung, Chin-chu. Positions of limit-cycles of the system

$$dx/dt=\sum_{0\le i+k\le 2} a_{ik}x^iy^k,\quad dy/dt=\sum_{0\le i+k\le 2} b_{ik}x^iy^k.$$

Sci. Sinica **8** (1959), 151–171.

It is known that the given system has at most three limit-cycles [references: N. N. Bautin, Mat. Sb. (N. S.) **30 (72)** (1952), 181–196; MR **13**, 652; I. G. Petrovskiĭ and E. M. Landis, ibid. **37 (79)** (1955), 209–250; MR **17**, 364]. Let C_i designate a limit-cycle and I_i the bounded component of the complement of C_i. In this paper the author establishes their possible relative positions: If there are two limit-cycles, then the two positions $I_1\cap I_2\neq 0$ and $I_1\cap I_2=0$, are possible; if there are three limit-cycles, then the two positions $I_i\cap I_j\neq 0 (i,j=1,2,3)$ and $I_1\cap I_2\neq 0$, $I_1\cap I_3=0$ are possible while the two positions $I_1\cap I_2=0$, $I_i\cap I_3\neq 0$ $(i=1,2)$ and $I_i\cap I_j=0$ $(i,j=1,2,3;\ i\neq j)$ are impossible. The proof consists of detailed study of the properties of the vector field defined by the system and construction of concrete examples.

Choy-tak Taam (Washington, D.C.)

5041:

Denjoy, Arnaud. Les équations différentielles périodiques. Points d'accumulation des intégrales. C. R. Acad. Sci. Paris **248** (1959), 497–500.

In this last [except for the afterthought, reviewed below] of a series of notes on differential equations on S^3, the three-torus, the author compares known facts about flows on S^2 with corresponding counterexamples and conjectures for the three-dimensional case. He points out that when there exist cycles, the rotation number cannot be defined in the usual way; discusses the limit sets in invariant subsets ("type (E)") which contain no cycles; and asks for a criterion on the differential equation similar to his classical result for inferring the ergodic case.

L. W. Green (Minneapolis, Minn.)

5042:

Denjoy, Arnaud. Les systèmes différentiels périodiques. Propriétés ergodiques et stabilité des trajectoires. C. R. Acad. Sci. Paris **248** (1959), 1253–1258.

This is a postscript to the author's series concerned with qualitative questions for the system $dX/dz=F(z,X)$ on S^3, the three-torus, where $X=(x,y)$ and F is a two-vector of period one in all its variables. A minimal set in this flow is called a skein (écheveau). A skein is either S^3 or nowhere dense. By building a flow under a function, an example of a skein in S^3 is given whose projection on a

surface of section is also a skein for a (different) two-dimensional flow. A "rotation vector" may be defined for certain kinds of skeins. A classification for types of skeins is discussed. *L. W. Green* (Minneapolis, Minn.)

5043:

Laitone, E. V. On the damped oscillations equation with variable coefficients. Quart. Appl. Math. **16** (1958), 90–93.

An estimate for the solutions of the differential equation

$$(1)\qquad u''+p(t)u'+q(t)u = 0$$

is obtained for the oscillatory case, i.e., for

$$\phi = (q-p^2/4-p'/2) \geqq m^2 > 0$$

under the additional assumption that ϕ is monotonic. For every solution of (1) there holds

$$(2)\quad |u(t)| \leqq m^{-1}[\phi(0)u^2(0)+(u'(0)+p(0)u(0)/2)^2]^{1/2} \times \exp\left(-\int_0^t p/2\,dt\right).$$

This inequality is obtained in transforming (1) in the well-known way into $v''+\phi v=0$ and using the auxiliary function $F=\phi v^2+v'^2$ that is often applied in connection with equation (1) [see E. Kamke, #5036]. An application to dynamic stability analysis of missiles, in particular to the case of vertical entry into the earth atmosphere at hypersonic speed, is discussed.

K. Matthies (Columbia, S.C.)

5044:

Erugin, A. N. The asymptotic representation of integral curves in the case of a focus. Dokl. Akad. Nauk BSSR **2** (1958), 234–237. (Russian)

The author studies the integral curves of a system

$$(1)\qquad x' = P(x,y),\quad y' = Q(x,y),$$

where P and Q are holomorphic at (0, 0). It is assumed that both roots of the characteristic equation of (1) at (0, 0) are pure imaginary, and that the origin is a focus. By changing to polar coordinates, and making an appropriate Lyapunov transformation (1) is transformed to

$$dr/d\theta = gr^m+\sum_1^\infty R_n(\theta)r^{m+n},$$

where $m\geqq 3$ is an odd integer, the R_n are polynomials in $\sin\theta$ and $\cos\theta$, and $g\neq 0$. When $g<0$ [$g>0$] $r\to 0$ as $\theta\to+\infty$ [$\theta\to-\infty$]. Assuming $g<0$, the author shows that

$$r = \theta_1^{-1}(1+\gamma(\theta)+o(\theta_1^{-2m+3})),$$

where $\theta_1^{m-1}=(1-m)g\theta$. An explicit approximation for $\gamma(\theta)$ is given, which involves many complicated formulas. No proofs are given. *W. S. Loud* (Madison, Wis.)

5045:

Tabueva, V. A. The form of the region of attraction of the null solution of a certain differential equation of second order. Mat. Sb. (N.S.) **47(89)** (1959), 209–220. (Russian)

The author considers the system

$$x' = y,\quad y' = -\phi(x)y-f(x),$$

where $\phi(x)$, $f(x)$ are continuously differentiable functions, periodic of the same period 2π and satisfying these conditions: $0<m\leqq\phi(x)$ for all x, $xf(x)>0$ near $x=0$, $f(0)=0$, and $f(x_i)=0$, $f'(x_i)\neq 0$, $i=1, 2$, where $x_1>0>x_2$ and x_2 is the zero of $f(x)$ nearest to $x=0$, $x_1=x_2+2\pi$. Such equations have been studied under similar assumptions before [cf., e.g., Amerio, Ann. Mat. Pura Appl. (4) **30** (1949), 75–90; Ann. Scuola Norm. Sup. Pisa (3) **3** (1950), 19–57; MR **11**, 723; **12**, 180; Seifert, Z. Angew. Math. Phys. **3** (1952), 468–471; MR **14**, 647; L. N. Belyustina, *Pamyati A. A. Andronova*, pp. 173–186, Izdat. Akad. Nauk SSSR, Moscow, 1955; Izv. Akad. Nauk SSSR. Otd. Teh. Nauk **10** (1954), 131–140; MR **17**, 370; **16**, 823. Here it is proved that if, in addition, $\int_x^{x+2\pi} f(x)dx<0$, then the trajectories can be of three and only three distinct qualitative types, which are called asynchronous, synchronous, and critical. For the first two types sufficient criteria are given in terms of inequalities involving certain integrals of $f(x)$ and $\phi(x)$.

H. A. Antosiewicz (Los Angeles, Calif.)

5046:

Lykova, O. B. On the investigations of individual solutions of a system of differential equations with a small parameter on a two-dimensional local integral manifold in the resonance case. Ukrain. Mat. Ž. **10** (1958), 365–374. (Russian. English summary)

In a previous paper [same Ž. **9** (1957), 419–431; MR **19**, 857], the present author has shown that if the system of equations (1) $dx/dt=X(x)$ has an asymptotically stable two-manifold M of periodic solutions, then the perturbed system (2) $dx/dt=X(x)+\varepsilon X(t,x)$, $X(t+2\pi,x)=X(t,x)$, has an asymptotically stable two-manifold M_ε of solutions for ε sufficiently small, and $M_\varepsilon\to M$ as $\varepsilon\to 0$. [See cited review for conditions on $X(x)$, $X(t,x)$.] Furthermore, on M_ε, the solutions of (2) satisfy a system of two first order equations (3)

$$da/dt = \varepsilon Q(t,\psi,a,\varepsilon),\quad d\psi/dt = \omega(a)+\varepsilon P(t,\psi,a,\varepsilon),$$

where P, Q are periodic in t, ψ of period 2π. The present paper discusses the behavior of the solutions of (3) by using a generalization of the method of averaging of N. Krylov and N. Bogolyubov [see the translation by S. Lefschetz, *Introduction to non-linear mechanics*, Ann. of Math. Studies no. 11, Princeton Univ. Press, 1943; MR **4**, 142] for the case where $\omega(a)$ is approximately a rational number, i.e. the case of resonance. The case of nonresonance was discussed previously [same Ž. **10** (1958), 239–250; MR **21** #2778].

J. K. Hale (Baltimore, Md.)

5047:

★Красовский, Н. Н. Некоторые задачи теории устойчивости движения. [Krasovskiĭ, N. N. Certain problems in the theory of stability of motion.] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959. 211 pp. 8.70 rubles.

This is one more highly interesting and valuable monograph on stability, and more strictly on Liapunov's direct method, its inversion, extensions and applications. The author is indeed one of the major and most original contributors to this general theory. The problems are constantly elucidated with clarity, the definitions are given in full and most proofs are dealt with completely unless they are standard and readily accessible—at least in the Russian literature.

Not only is ordinary stability dealt with but also stability over a finite time interval—practical stability. Furthermore, constant attention is paid to the size of the domain of asymptotic stability—another practical consideration. For it stands to reason that if this domain is minute what takes place is practical instability. An extensive bibliography of 141 titles (11 non-Soviet titles) terminates the monograph.

A bird's-eye view chapter by chapter follows.

Short Introduction describing the problems to be discussed.

Chapter 1: Existence theorems for the scalar functions $v(x, t)$ (x is an n-vector) satisfying the conditions of the theorems of Liapunov. Given the n-vector equation (*) $\dot{x}=X(x, t)$, $X(0, t)=0$ for $t\geqq 0$, one considers its stability behavior at the origin. There are four classical theorems due to Liapunov giving sufficient conditions for stability, for asymptotic stability and for instability (two theorems). The last two theorems are special cases of a broader instability proposition due to Četaev. In all these theorems there figure the fixed sign in the neighborhood of the origin of a certain function $v(x, t)$ and the fixed sign (or zero) of its time derivative $\dot{v}(x(t), t)=\partial v/\partial t+\operatorname{grad} v\cdot X$ (derivative along the solutions of (*)). The chapter deals with the direct Liapunov theorems and with their inversion. Especially noteworthy is the author's equivalent condition for the existence of a $v(>0, 0$ at the origin) bounded from above by a $w(x)$ (behaving like v) and such that v is of fixed sign or zero. It may be formulated approximately thus (Property A): Let $S(\alpha)$, $S(\alpha, \beta)$ denote the sets $\|x\|<\alpha$, $\alpha\leqq\|x\|\leqq\beta$ and let (*) operate in $S(A)$. Given $0<\eta<\varepsilon<A$, and t_0, there exists a time $T(\varepsilon, \eta)$ such that if $t_0\geqq T$ and the trajectory x starts at x_0 in $S(\eta, A)$ then its segment in $[t_0-T, t_0+T]$ does not lie entirely in $S(\varepsilon)$. For autonomous systems (X is a function of x alone: steady state) one may substitute for A this beautiful property: some neighborhood of the origin must be free from complete trajectories $(-\infty<t<+\infty)$ of the system.

Chapter 2: Certain modifications of the theorems of Liapunov. The Liapunov statements admit of tightening up in the direction of various uniformities. These are dealt with particularly as they are related to the properties of the functions v.

Chapter 3: Certain generalizations of the theorems of Liapunov. Especially noteworthy features are: (a) weakening of the Liapunov criteria for asymptotic stability and instability; (b) tying up the performance of a function $v(x, t)$ (of Liapunov type) with the local performance of such functions for neighborhoods of every trajectory (near the origin); (c) a discussion (not too conclusive) of systems (*) where $X(x, t)$ has discontinuities. There is notable work done recently on this topic by Aĭzerman and Gantmaher [Prikl. Mat. Meh. **21** (1957), 658–669; MR **20** #153b].

Chapters 4 and 5: Various applications of the Liapunov doctrines. Given (*) and (**) $\dot{x}=X+R(x, t)$, if (*) has certain stability properties what variation in $\|R\|$ may be allowed without losing these properties? This is the central theme of Chapter 4. Many noteworthy questions are related to this problem. Of particular interest are: (a) determination of the structural stability (what the Russians call "rough character") of such and such stability property; (b) assuming that (*) operates for all t, x and that the Jacobian matrix $\partial X/\partial x$ is bounded, a discussion of suitable linear approximations to the system.

Various related problems are discussed in Chapter 5: Stability where $\|R\|$ in (**) is bounded in the mean (rapid oscillations even of sizable amplitude do not affect stability). Control problems such as are dealt with by A. M. Letov, *Ustoĭčivost' nelineĭnyh reguliruemyh sistem* [Izdat. Tehn.-Teor. Lit., Moscow, 1955; MR **17**, 487]. Critical cases (some characteristic roots zero, the rest with negative real parts), treated more or less in the style of I. G. Malkin, *Teoriya ustoĭčivosti dviženiya* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952; translated as *Theorie der Stabilität einer Bewegung*, Oldenbourg, Munich, 1959; MR **15**, 873; **21** #2791].

Chapters 6 and 7: Extension of the Liapunov theory to systems with time lag. The systems considered are of the type $\dot{x}=X(x(t+\theta), t)$, where x, X are n-vectors, X being a functional $X(x(\theta), t)$, where $\theta\in[-h, 0]$, $h>0$. A careful study is made of these systems, definitions of stability given and the theorems of Liapunov extended to them. The natural applications which suggest themselves are discussed. *S. Lefschetz* (Princeton, N.J.)

5048:

★**Hahn, Wolfgang. Theorie und Anwendung der direkten Methode von Ljapunov.** Ergebnisse der Mathematik und ihrer Grenzgebiete. N. F., Heft 22. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. vii+142 pp. DM 28.00.

The direct method or, as Lyapunov called it in his memoir [*Problème général de la stabilité du mouvement*, Princeton Univ. Press, 1949; MR **9**, 37], the second method, embodies all those criteria for the stability and instability of a solution of an equation $x'=f(t, x)$ that have one feature in common: they are based solely upon properties of scalar functions $V(t, x)$ and their total derivative $\partial V/\partial t+\operatorname{grad} V\cdot f$ and do not depend upon considerations of variational equations and the like. Lyapunov's original three basic theorems (one on stability, two on instability) and one remark (on asymptotic stability), formulated in terms of such functions, are the foundation of essentially all later work, much of it done in the USSR. The present book gives a summary account of the development in this area of stability theory up to 1958. It should aid greatly in making the second method more widely available.

Chapter I deals with the basic definitions of stability and instability. In II there are taken up Lyapunov's theorems on stability (in their original formulation) and instability and Četaev's instability theorem. Applications to linear and nearly linear equations and to the problems of Aĭzerman and of Lur'e are given in III. Uniform stability and uniform asymptotic stability and the various converse theorems are treated in IV. Lyapunov functions with particular growth properties, order numbers for linear equations, exponential stability are discussed in V. In VI stability in the first approximation and total stability are taken up. VII deals with the critical cases, particularly with the results of Malkin for special cases. In VIII generalizations to dynamical systems in general metric spaces are briefly given, and applications to differential-difference and difference equations are discussed. The bibliography is excellent. Format and printing are of the high standard common to Springer books. *H. A. Antosiewicz* (Los Angeles, Calif.)

5049:

Livartovskii, I. V. Some tests for the stability of the solutions of a system of differential equations with discontinuous right-hand members. Dokl. Akad. Nauk SSSR **125** (1959), 733–736. (Russian)

Aĭzerman and Gantmaher [same Dokl. **116** (1957), 527–530; Prikl. Mat. Meh. **21**(1957), 658–669; MR **20** #153ab] proved criteria for the stability of a periodic solution of $x'=f(x,t)$ with f discontinuous and periodic in t. The present author considers the same problem for an arbitrary solution with f discontinuous but not necessarily periodic in t. Under hypotheses too lengthy to quote he states theorems for the asymptotic stability, in terms of the asymptotic behavior of a fundamental matrix of the linear approximation and the existence of a Lyapunov function, which parallel the known results for the continuous case. *H. A. Antosiewicz* (Los Angeles, Calif.)

5050:

Cherry, T. M. The pathology of differential equations. J. Austral. Math. Soc. **1** (1959/61), part 1, 1–16.

This paper is an expository account (presidential address delivered in 1957) on several questions on the behavior of nonlinear differential equations in the large. Denjoy's [J. Math. Pures Appl. **11** (1932), 333–375] results on the flow on a torus are discussed, and E. Artin's [Abh. Math. Sem. Univ. Hamburg **3** (1924), 170–175] and M. Morse's [Trans. Amer. Math. Soc. **22** (1921), 84–100] theorems on transitivity of the geodesic flow on a 2-dimensional surface with negative curvature. Littlewood's [Acta. Math. **97** (1957), 267–308; MR **19**, 548] investigation on van der Pol's equation with forcing terms is mentioned as well as C. L. Siegel's [Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1952**, 21–30; MR **15**, 222] description of the solutions of an analytic system near an equilibrium by convergent series. An interesting example of a nonintegrable Hamiltonian system discussed in Section 6 seems to be new. *J. Moser* (Cambridge, Mass.)

5051:

Šimanov, S. N. Almost periodic oscillations in nonlinear systems with retardation. Dokl. Akad. Nauk SSSR **125** (1959), 1203–1206. (Russian)

The author considers the equation

$$\dot{x} = \sum_{1}^{k} A_j x(t-\tau_j) + X(t, x(t-\tau_1), \cdots, x(t-\tau_k)) + \varepsilon F(t, x(t-\tau_1), \cdots, x(t-\tau_k), \varepsilon)$$

under the following assumptions: (i) $\tau_1=0$, $\tau_2, \cdots, \tau_k$ are positive constants, and the A_j are matrices such that $|\sum A_j \exp(-\lambda\tau_j)-\lambda I|=0$ has solutions λ_i, Re $\lambda_i<-2\alpha<0$; (ii) $F(t, x_1, \cdots, x_k, \varepsilon)$ is almost periodic in t uniformly in x_j for $|x_j|\leq R$ and in ε for $|\varepsilon|\leq\varepsilon_1$; (iii) F is lipschitzian in x_j and continuous in ε; (iv) $X(t, x_1, \cdots, x_k)$ is lipschitzian in x_j for a function $q(R)>0$ where $q(R)\to 0$ with R. He shows that there exist positive ε^*, R^* [$\varepsilon^*(R)$] such that for every ε with $|\varepsilon|<\varepsilon^*$ the equation [with $X\equiv 0$] has a unique almost periodic solution in $|x|<R^*$ [$|x|<R$]. The proofs are based on the lemma proved in detail that $\dot{x}=\sum A_j x(t-\tau_j)+f(t)$ with f almost periodic and $|f|<M$ has a unique almost periodic solution $x^*(t)$ with $|x^*(t)|<MA$ where A depends solely on A_j and τ_j. *H. A. Antosiewicz* (Los Angeles, Calif.)

5052:

Norkin, S. B. Solutions of a linear homogeneous differential equation of second order with lagging argument. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 199–206. (Russian)

The author considers the second order functional equation

$$u''(x)+p(x)u(x)+q(x)u(x-r(x)) = 0,$$

and shows that under appropriate conditions every solution is a linear combination of two particular solutions. In addition, some results concerning the oscillation of the solutions are given. *R. Bellman* (Santa Monica, Calif.)

5053:

Krasovskii, N. N. On the theory of optimal regulation of non-linear systems of second order. Dokl. Akad. Nauk SSSR **126** (1959), 267–270. (Russian)

The author considers the problem of optimal control for a system $\dot{x}=f(x,t)+q(t)\eta$, where f, q are given 2-vectors continuous for all x and $t\in(t_\alpha, t_{\alpha+1})$, $\{t_\alpha\}$ being an increasing sequence; it is assumed that f has continuous uniformly bounded partials $\partial f_i/\partial x_j$ for $t\in(t_\alpha, t_{\alpha+1})$ and that f and $\partial f_i/\partial x_j$ have only discontinuities of the first kind at $t=t_\alpha$. Among the results he announces are the following. If for a given x_0 there exists at least one control $\eta(t)$ with $|\eta(t)|\leq 1$ such that the solution, $x(x_0, t, \eta)$, through $(x_0, 0)$ vanishes at some $t=T$, then there exists an optimal solution $x(x_0, t, \eta_0)$ vanishing at $t=T_0\leq T$ which corresponds to a piecewise continuous control $\eta_0(t)$ with $|\eta_0(t)|\leq 1$. The optimal control is given by $\eta_0(t)=\operatorname{sign}(l_0\cdot h(T_0, t))$ where $l_0\neq 0$ is some constant vector and $h(t,s)=F(t)F^{-1}(s)q(s)$, $F(t)$ being a fundamental matrix of the variation equation $\dot{z}=(\partial f_i/\partial x_j)z$ along $x(x_0, t, \eta_0)$. *H. A. Antosiewicz* (Los Angeles, Calif.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 5036, 5137, 5142a-b, 5219, 5240, 5241, 5242.

5054:

Bicadze, A. V. Some linear problems for linear partial differential equations. Advancement in Math. **4** (1958), 321–403. (Chinese)

This is the Chinese translation of a part of the author's lecture notes given at Peking. The lectures are expository and intended as introductions to the general theory of elliptic, hyperbolic, parabolic and mixed type differential equations. Theorems of uniqueness, existence and statements on well proposed problems are explained mostly for equations of second order in two independent variables. Higher order equations with more than two variables are occasionally treated. Applications of linear operators are briefly mentioned at the end. *Yu Why Chen* (New York, N.Y.)

5055:

Pliś, A. On contact points of two integrals of a partial differential equation of the first order. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 11–14.

If two integrals $z(x,y)$ and $s(x,y)$ of the equation

$z_x=f(x, y_1, \cdots, y_n, z, z_{y_1}, \cdots, z_{y_n})$ are tangent at a point, then they are tangent along the characteristic through that point, provided the functions f, z, s are all of class C^2. In this paper, the same result is proved under the following more general assumptions: (a) For $n=1$, the functions z, s, f are of class C^1 and there is a unique solution of the characteristic equations $y'=-f_y\ (x, y, z, q)$, $z'=f-qf_q$, $q'=f_y+qf_z$ passing through the point $(0, 0, c, k)$, a point such that f is of class C^1 in a neighborhood of it. (b) For $n>1$, f is of class C^2 and z and s are of class C^1 and satisfy inequalities of the form

$$|z_{y_i}(x, y_1, \cdots, y_n)-z_{y_i}(x, \bar{y}_1, \cdots, \bar{y}_n)| \leqq \sum |y_i-\bar{y}_i|^s,$$

where $s>\frac{1}{2}$, in the neighborhood of the origin. The author also gives an example to show that, for $n=1$, it is not sufficient to assume merely that the functions z, s, f are of class C^1. *D. L. Bernstein* (Towson, Md.)

5056:

Gorbunov, A. D.; and Budak, B. M. The method of straight lines for solution of a non-linear boundary problem in a region with curvilinear boundary. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 3, 3–12. (Russian)

Let $\bar{G}=\{(x, y)|\ 0\leqq x\leqq l_x,\ g(x)\leqq y\leqq l_y\}$, where $g(0)=0$, $g'(x)\geqq 0$ for $0\leqq x\leqq l_x$. The non-linear boundary value problem

$$(1)\qquad \begin{aligned} &u_{xy} = f(x, y, u, u_x, u_y), \quad (x, y) \in \bar{G},\\ &u(x, g(x)) = \varphi(x), \quad 0 \leqq x \leqq l_x,\\ &u(0, y) = \psi(y), \quad 0 \leqq y \leqq l_y \end{aligned}$$

is replaced by the system of ordinary differential equations

$$(2)\qquad \begin{aligned} &u'_{k+1}(y)-u'_k(y) =\\ &\quad hf(x_k, y, u_k(y)), h^{-1}(u_{k+1}(y)-u_k(y)), u'_k(y)),\\ &u_0(y) = \psi(y), \quad u_k(g(x_k)) = \varphi(x_k), \quad k = 1, 2, \cdots \end{aligned}$$

for the functions $u_k(y)$ (intended as approximations to $u(x_k, y)$) where $x_k=kh$. It is shown that under suitable Lipschitz conditions on f a solution of (2) exists for $(x_k, y) \in \bar{G}$. Without assuming the existence of a solution of (1) it is then shown that the functions $\tilde{u}(x, y)$ obtained by linear interpolation between $u_k(y)$ and $u_{k+1}(y)$ $(k=0, 1, \cdots)$ converge for $h\to 0$ to a solution $u(x, y)$ of (1) in a suitable subdomain of $\bar{G}$, and that this solution is unique. Estimates for the error $|\tilde{u}(x, y)-u(x, y)|$ are also given. Similar statements are made for the derivatives $\tilde{u}_x(x, y)$ and $\tilde{u}_y(x, y)$. *P. Henrici* (Los Angeles, Calif.)

5057:

Agmon, Shmuel. Multiple layer potentials and the Dirichlet problem for higher order elliptic equations in the plane. I. Comm. Pure Appl. Math **10** (1957), 179–239.

Soit Ω un ouvert de R^n; soit Γ la frontière de Ω. Soit A un opérateur elliptique d'ordre $2m$. On considère le problème de Dirichlet pour l'opérateur A dans l'ouvert Ω (1) $Au|\Omega=0$, $D^pu|\Gamma=f^p(|p|\leqq m-1)$. Pendant les derniers dix ans plusieurs auteurs (Višik, Gårding, Browder, Lions, ...) ont étudié le problème de Dirichlet dans le cas $m=$quelconque en utilisant la théorie des espaces Hilbertiens. Cependant, dans le cas $m=1$ et, en particulier, dans le cas de l'opérateur de Laplace, il y a classiquement une autre méthode, la méthode de Neumann-Poincaré-Fredholm [voir, par exemple, Miranda, *Equazioni alle derivate parziali di tipo ellittico*, Springer, Berlin-Göttingen-Heidelberg, 1955; MR **19**, 421]. Dans cet article l'A. généralise cette méthode au cas $m=$quelconque, $n=2$, A homogène à coefficients constants, Ω relativement compact et simplement connexe. Il dit que la théorie ne va pas dans le cas $n>2$ parce que, en général, les "noyaux de Poisson" ne s'expriment pas comme une somme des dérivées de la solution élémentaire distinguée, ce qui est décisif dans la déduction. (Une autre généralisation, dans le cas le plus général, a été donnée par Lopatinskiĭ [Ukrain. Mat. Ž. **5** (1953), 123–151; MR **17**, 494], mais la théorie de l'A. semble être plus proche à la forme classique.) Citons ici seulement le résultat final: Théorème. Supposons que Ω appartienne à la classe $C_\beta(\beta>\frac{1}{2})$ et que les f^p soient les restrictions à Γ des dérivées d'une fonction dans la classe $C_{(m-1)+\alpha}(\alpha<\beta)$ dans Ω. Alors, il existe une solution et une seule de (1) et cette solution u appartient à la classe $C_{(m-1)+\alpha}$. *J. Peetre* (Zbl **81**, 98)

5058:

Boyarskiĭ, B. V. Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients. Mat. Sb. N.S. **43(85)** 1957, 451–503. (Russian)

A uniformly elliptic system $v_y=\alpha u_x+\beta u_y+au+bv+e$, $-v_x=\gamma u_x+\delta u_y+cu+dv+f$ given in a bounded plane domain G can be rewritten in the form (*) $w_{\bar{z}}-q_1(z)w_z-q_2(z)\overline{w}_{\bar{z}}=Aw+B\overline{w}+C$, where $w=u+iv$. The uniform ellipticity implies $|q_1|+|q_2|\leqq q_0<1$, where q_0 is a constant. The functions q_1 and q_2 are assumed measurable complex-valued functions of z, and $A, B, C \in L_p(G)$, where $p>2$ is a given number.

In this generality the above equations were treated by L. Bers and L. Nirenberg [*Convegno internazionale sulle equazioni lineari alle derivate parziali*, Trieste, 1954, pp. 111–140, Edizioni Cremonese, Rome, 1955; MR **17**, 974]. A correction to a misprint in the Bers-Nirenberg paper was inserted at the end of the volume, after the index. This was overlooked by the present author who was mistakenly led to challenge some of their proofs. Actually the most important results in the present paper are already in the Bers-Nirenberg paper. However, the methods used in the present paper are entirely different and are independently interesting. They are based on the Calderón-Zygmund inequality [Acta Math. **88** (1952), 85–139; MR **14**, 637] and are independent of the theory of uniformization.

A generalized regular solution $w(z)$ of (*) is (shown to be) a function which is continuous in G and possesses generalized L_p derivatives w_z and $w_{\bar{z}}$ in G, i.e., $w(z) \in W_p(G)$. The functions w_z, $w_{\bar{z}}$ and w almost everywhere in G satisfy (*). As a sample of the results we quote: To each entire function $f(z)$ there corresponds a unique generalized regular solution $w(z)$ such that $w(z)$ is continuous in the entire plane E, analytic in the complement of $\bar{G}$, $w(z) \in W_p(E)$, and $w(z)\sim f(z)$ for $z\to\infty$. Method: A solution is sought in the form

$$w(z) = f(z)-(1/\pi) \iint \omega(t)/(t-z)dG = f(z)+T(\omega).$$

It is readily shown that ω must satisfy an equation of the form

$$\omega-q_1\cdot S(\omega)-q_2\cdot\overline{S(\omega)} = AT(\omega)+B\overline{T(\omega)}+C_0,$$

where C_0 is known and 0 outside G, and

$$S(\omega) = -(1/\pi)\iint \omega(t)/(t-z)^2 dE.$$

S can be estimated by the Calderón-Zygmund inequality and the problem is ultimately reduced to a fixed point theorem. See also MR **17**, 157.

A. N. Milgram (Berkeley, Calif.)

5059:

Slobodeckiĭ, L. N. S. L. Sobolev's spaces of fractional order and their application to boundary problems for partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 243–246. (Russian)

L'A. définit les espaces de Sobolev d'ordre fractionnaire pour l'espace L^2 (i.e., l'espace des fonctions de carré sommable sur un ouvert ainsi qu'un certain nombre de leurs dérivées, usuelles ou fractionnaires). Théorèmes de trace et applications à quelques problèmes aux limites de nature elliptique et parabolique. *J. L. Lions* (Nancy)

5060:

Slobodeckiĭ, L. N. Estimates of solutions of elliptic and parabolic systems. Dokl. Akad. Nauk SSSR **120** (1958), 468–471. (Russian)

The author takes his departure from the paper reviewed above, in which he has defined certain function spaces in terms of the summability of the elements, and norms in terms of the integrals. He adopts Petrovskiĭ's definition of an elliptic operator [Uspehi Mat. Nauk (N. S.) **1** (1946), no. 3-4 (13–14), 44–70; MR **10**, 301]. In these terms he states five theorems without proof relating the norms of the elements of the space and those of their transforms by the operators considered.

A. S. Householder (Oak Ridge, Tenn.)

5061:

Slobodeckiĭ, L. N. Estimates in L_p of solutions of elliptic systems. Dokl. Akad. Nauk SSSR **123** (1958), 616–619. (Russian)

Let Ω be a bounded domain in Euclidean n-space E_n with boundary S of Ω a $(q+1)$-times continuously differentiable manifold. Let $v=v(x)\in W_p^{(q)}(\Omega)$. The author proves the following extension of results previously obtained by himself [#5059 above] and E. Gagliardo [Rend. Sem. Mat. Univ. Padova **27** (1957), 284–305; MR **21** #1525]: The normal derivatives $\partial^k v/\partial \nu^k$ ($k=0, 1, \cdots, q-1$), considered as functions of points of S, belong to the space $W_p^{(q-k-1/p)}(S)$, and

$$\|\partial^k v/\partial\nu^k\|_{W_p^{(q-k-1/p)}(S)} \leqq C_1\|v\|_{W_p^{(q)}(\Omega)}.$$

Conversely, given functions $\varphi_k(x')\in W_p^{(q-k-1/p)}(S)$, there exists a function $v\in W_p^{(q)}(\Omega)$ satisfying $\partial^k v/\partial\nu^k=\varphi_k$ on S, and

$$\|v\|_{W_p^{(q)}(\Omega)} \leqq C_2 \sum \|\varphi_k\|_{W_p^{(q-k-1/p)}(S)}.$$

These results are then applied to elliptic operators $L(x, \partial/\partial x)$ of order $2m$ in $\Omega+S$. Let $R_\mu(x', \partial/\partial x)$ be differential operators of orders $m_\mu(\mu=1, \cdots, m)$ suitably related to L [#5059], let $q\geqq 2m$ and suppose the coefficients of L and R_μ to admit $q-2m$ bounded derivatives; then for any $u(x)\in W_p^{(q)}(\Omega)$ we have

$$C_1[\|Lu\|_{W_p^{(q-2m)}(\Omega)}+\sum\|R_\mu u\|_{W_p^{(q-m_\mu-1/p)}(S)} \leqq \|u\|_{W_p^{(q)}(\Omega)}] \leqq C_2[\|Lu\|_{W_p^{(q-2m)}(\Omega)}+\sum\|R_\mu u\|_{W_p^{(q-m_\mu-1/p)}(S)}+\|u\|_{L_p(\Omega)}].$$

The note also contains a discussion of extensions of the preceding to unbounded domains.

A. N. Milgram (Berkeley, Calif.)

5062:

Fichera, Gaetano. Una introduzione alla teoria delle equazioni integrali singolari. Rend. Mat. e Appl. (5) **17** (1958), 82–191.

Dans un domaine B de R^2 on considère l'opérateur différentiel elliptique

$$\mathscr{E} = \frac{\partial}{\partial x}\left(a_{11}\frac{\partial}{\partial x}+a_{12}\frac{\partial}{\partial y}\right)+\frac{\partial}{\partial y}\left(a_{21}\frac{\partial}{\partial x}+a_{22}\frac{\partial}{\partial y}\right),$$

où $a_{ij}\in\mathscr{C}_h^2$ (une fonction appartient à $\mathscr{C}_h^m$ si elle est m fois continûment dérivable, ses dérivées d'ordre m vérifiant une condition de Hölder uniforme, d'ordre h); $a_{11}\lambda_1^2+2a_{12}\lambda_1\lambda_2+a_{22}\lambda_2^2$ est définie positive, $a_{11}a_{22}-a_{12}^2=1$. Soit A borné $\subset B$, de frontière Σ simple fermée de classe $\mathscr{C}_h^1$. Soient α et $\beta\in\mathscr{C}^1(A)\cap\mathscr{C}^0(\bar{A})$ avec $a_{11}\alpha_x+a_{12}\alpha_y-\beta_y=0$, $a_{21}\alpha_x+a_{22}\alpha_y+\beta_x=0$ (on dit que α et β sont $\mathscr{E}$ conjuguées). Alors si $\beta(z_0)=0$, et $p>1$,

$$(1)\qquad \|\beta\|_{L^p(\Sigma)} \leqq K_p(A, z_0)\|\alpha\|_{L^p(\Sigma)}$$

généralisation d'un résultat classique de M. Riesz [Math. Z. **27** (1927), 218–244]. Soit $u\in\mathscr{C}^2(A)\cap\mathscr{C}^1(\bar{A})$ avec $\mathscr{E}u=0$; on a: $\int_\Sigma \alpha(\partial u/\partial\nu)ds=\int_\Sigma\beta(\partial u/\partial s)ds$; comme pour $\varphi\in\mathscr{C}_k^1(\Sigma)$ existent α et β, $\mathscr{E}$ conjuguées, avec $\alpha|_\Sigma=\varphi$, on a:

$$\left\|\frac{\partial u}{\partial\nu}\right\|_{L^q(\Sigma)} = \sup_\alpha\left|\int\alpha\frac{\partial u}{\partial\nu}ds\right|\Big/\|\alpha\|_{L^p(\Sigma)}\quad(1/p+1/q=1),$$

de sorte que (1) entraîne

$$(2)\qquad \left\|\frac{\partial u}{\partial\nu}\right\|_{L^q(\Sigma)} \leqq K_p\left\|\frac{\partial u}{\partial s}\right\|_{L^q(\Sigma)}.$$

Comme $\int_\Sigma\alpha(\partial u/\partial s)ds=-\int_\Sigma\beta(\partial u/\partial\nu)ds$, on a de même

$$(2')\qquad \left\|\frac{\partial u}{\partial s}\right\|_{L^q(\Sigma)} \leqq K_p\left\|\frac{\partial u}{\partial\nu}\right\|_{L^q(\Sigma)}.$$

L'A. retrouve ensuite les résultats connus pour le problème de dérivée oblique [cf. avec des hypothèses moins fortes, I. N. Vekua, Dokl. Akad. Nauk SSSR **92** (1953), 1113–1116; MR **15**, 798].

Sur Σ comme ci-dessus, on considère l'opérateur $\varphi\to\mathscr{S}\varphi=p\varphi-q\mathscr{K}\varphi+\mathscr{M}\varphi$, où $\varphi\in\mathscr{C}_{k(\varphi)}^0$, $0<k(\varphi)\leqq 1$; p et $q\in\mathscr{C}_k^0$, à valeurs complexes; $\mathscr{M}$ est un opérateur intégral complètement continu de $L^p(\Sigma)$ dans lui-même; $\mathscr{K}\varphi=\pi^{-1}\int_\Sigma K(z,\zeta)\varphi(\zeta)ds_\zeta$, K étant un noyau singulier de type elliptique (pour cette définition, cf. le mémoire, p. 89). Utilisant une forme canonique de $\mathscr{S}$ et (2), (2'), l'A. montre que $\mathscr{S}$ est continu de $L^p(\Sigma)$ dans lui-même, $p>1$. Si $p^2+q^2\neq 0$ sur Σ, $\mathscr{S}$ est réductible dans $L^p(\Sigma)$; de façon générale, si B est un espace de Banach et $\mathscr{S}\in\mathscr{L}(B;B)$, on dit que $\mathscr{S}$ est réductible s'il existe $\mathscr{S}'\in\mathscr{L}(B;B)$ avec $\mathscr{S}'\mathscr{S}=I+C$, I = identité, C compact. Si B est un espace de Hilbert, cette situation a été étudiée par S. G. Mihlin [à la bibliographie de l'A. il faut ajouter: S. G. Mihlin, Dokl. Akad. Nauk SSSR **57** (1947), 11–12; MR **9**, 241]; les résultats de Mihlin s'étendent. {Cette extension a été faite indépendamment par Mihlin [ibid. **125** (1959), 737–739; MR **21** #2186].} Applications.

J. L. Lions (Nancy)

5063:

Malgrange, Bernard. Sur une classe d'opérateurs dif-

férentiels hypoelliptiques. Bull. Soc. Math. France **85** (1957), 283–306.

The author's object is to find a suitable class of linear partial differential operators $P(X, D)$ with C^∞ coefficients which are hypoelliptic, that is, if T is a distribution such that $P(x, D)T$ is an indefinitely differentiable function, then T is an indefinitely differentiable function. In case $P(x, D) = P(D)$ has constant coefficients, then the complete class of such operators was found by Hörmander [(*), #5064 below] and the reviewer [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 39–41; MR **17**, 854]. The author's result is that if for each x_0 the constant coefficient operators $P(x_0, D)$ attached to x_0 are hypoelliptic, and if the various $P(x_0, D)$ are "equivalent", then $P(x, D)$ is hypoelliptic. This result was also derived independently by Hörmander [#5064] and Trèves.

The author also proves an inequality for these operators from which it follows that one can solve an analog of the Dirichlet problem. *L. Ehrenpreis* (Waltham, Mass.)

5064:

Hörmander, Lars. On interior regularity of the solutions of partial differential equations. Comm. Pure Appl. Math. **11** (1958), 197–218.

Nous dirons qu'un opérateur différentiel $P(x, D)$ (à coefficients indéfiniment différentiables) est hypoelliptique, si toute distribution solution u vérifiant (dans un ouvert) $P(x, D)u = f$ est une fonction indéfiniment différentiable là où f l'est. Dans le cas où P est un opérateur à coefficients constants, la caractérisation de tels opérateurs a été montrée par l'auteur (*) [L. Hörmander, Acta Math. **94** (1955), 161–248; MR **17**, 853]. En supposant ici les coefficients variables, l'A. montre une large classe d'opérateurs hypo-elliptiques que voici: (1) Pour tout point $x_0 \in \Omega$, $P(x_0, D)$ (opérateur tangentiel attaché au point x_0) est hypo-elliptique. (2) Soient x_0 et x_1 deux points quelconques de Ω, alors $P(x_0, D)$ et $P(x_1, D)$ sont également forts, c'est-à-dire que

$$1/K < (1+|P(x_0, i\xi)|)/(1+|P(x_1, i\xi)|) < K$$

pour tout ξ réel. Cette classe contient les opérateurs elliptiques et p-paraboliques au sens de Petrowsky.

Soit Ω un ouvert borné. L'A. utilise la norme suivante (utilisée par Morrey et Nirenberg): $N_{\Omega,\mu}(f) = \sup_{\delta>0} \delta^\mu \|f\|_{\Omega_\delta}$, où $\|f\|_{\Omega_\delta}$ désigne la L^2-norme de f sur Ω_δ, $f \in L^2_{\text{loc}}(\Omega)$ et Ω_δ désigne l'ensemble des points x de Ω vérifiant $\operatorname{dis}(x, \dot{\Omega}) < \delta$. Il montre l'inégalité:

$$N_{\Omega,\rho}(Q(D)u) \leq C(N_{\Omega,\gamma}(P(x, D)u) + N_{\Omega,0}(u)),$$

où $Q(D)$ est un opérateur plus faible que $P(x, D)$, et ρ et γ sont des constantes ≥ 0 ne dépendant que de P et Q. En utilisant cette inégalité, il montre d'abord l'hypoellipticité sous l'hypothèse que u est une solution ordinaire. Ensuite, la démonstration se fait en supposant que u est distribution. Il utilise ici une variante de la méthode de régularisation due à Friedrichs.

Cet article est écrit d'une manière claire et élégante, et les méthodes et des résultats intermédiaires pourront servir d'un fondement pour les recherches sur les opérateurs à coefficients variables. Finalement, on devrait ajouter que, bien que la classe qu'il a traitée soit assez large, cette classe n'est pas invariante pour le changement des variables indépendantes. *S. Mizohata* (New York, N.Y.)

5065:

Gårding, Lars; et Malgrange, Bernard. Opérateurs différentiels partiellement hypoelliptiques. C. R. Acad. Sci. Paris **247** (1958), 2083–2085.

The authors establish necessary and sufficient conditions for a partial differential operator with constant coefficients in R^{m+n} to be hypoelliptic with respect to the first m variables. Let x and y be coordinates in R^m and R^n, respectively, and denote derivatives with respect to $x\,[y]$ by $D_x\,[D_y]$. A distribution f in an open set $\Omega \subseteq R^{m+n}$ is called infinitely differentiable in x if for each open set $V_x \times W_y \subset \Omega$ and function $\varphi \in \mathscr{D}(W_y)$ (the set of infinitely differentiable functions with compact support in W_y) the distribution $\int f(x, y)\varphi(y)dy$ is infinitely differentiable in V_x. The partial differential operator with constant coefficients $P(D_x, D_y)$ is hypoelliptic in x if every solution f of $Pf = 0$ is infinitely differentiable in x. It is proved that for P to be hypoelliptic in x, it is necessary and sufficient that it have the form

$$P_0(D_x) + \sum P_j(D_x)Q_j(D_y) \quad (j > 0),$$

where P_0 is hypoelliptic and the P_j are strictly weaker than P_0 (i.e., $P_j(\xi)/P_0(\xi) \to 0$ as $\xi \to 0$ for $\xi \in R^m$). The necessity is proved by means of the closed graph theorem generalizing a method due to Hörmander [(*), #5064]. The sufficiency proof extends Fourier transform techniques of Malgrange [#5063] and Hörmander [#5064]. *M. Schechter* (New York, N.Y.)

5066:

Huet, Denise. Équations vectorielles elliptiques. C. R. Acad. Sci. Paris **247** (1958), 2085–2087.

Let $E\,[F]$ be a locally convex, complete, Hausdorff vector space with topology defined by a set $\Gamma_E\,[\Gamma_F]$ of semi-norms. Let $\mathscr{L}_b(E; F)$ be the space of continuous linear maps of E into F under the topology of bounded convergence. The author considers partial differential operators in R^n of the form $D = \sum_{|j| \leq m} \alpha_j(x) D^j$ (where $(j_1, j_2, \cdots, j_n)$, $|j| = j_1 + \cdots + j_n$, $D^j = \partial^{|j|}/\partial x_1^{j_1} \cdots \partial x_n^{j_n}$) having infinitely differentiable coefficients with values in $\mathscr{L}_b(E; F)$. Such operators map distributions (on R^n) with values in E into distributions with values in F. The operator D is called hypo-elliptic if for every open set $\Omega \subseteq R^n$, every distribution T, locally of finite order with values in E, is infinitely differentiable whenever DT is. It is proved that D is hypo-elliptic under the following assumptions. (1) D is elliptic, i.e., for each point $x_0 \in R^n$ and each semi-norm $p \in \Gamma_E$ there is a $q \in \Gamma_F$ such that $p(e) \leq q(\sum_{|j|=m} \alpha_j(x_0)\xi^j e)$ for all $e \in E$ and all $\xi \in R^n$ satisfying $|\xi| = 1$. (2) For every index r and every compact set $K \subset R^n$ having x_0 as an interior point and every index j satisfying $|j| = m$, there is a constant C such that $q(D^r(\alpha_j(x) - \alpha_j(x_0))e) \leq Cp(e)$ for all $e \in E$ and all $x \in K$. (3) For every index r and every compact set $K \subset R^n$ the set $\bigcup_{x \in K} D^r(\alpha(x))$ is equicontinuous in $\mathscr{L}(E; F)$.

It is noted that when E and F are Banach spaces (3) is automatically satisfied and that every elliptic operator satisfies (2). Other particular examples are cited including a special case of the result in the paper reviewed above. *M. Schechter* (New York, N.Y.)

5067:

Hörmander, Lars. Definitions of maximal differential operators. Ark. Mat. **3** (1958), 501–504.

Given any partial differential operator $P(D)$ with

constant coefficients and an open set Ω in R^ν, one can define, among other things, the following two extensions of $P(D)$ [cf. (*), #5064]: (i) The maximal operator P_ω (or weak extension) defined for those $u \in L^2(\Omega)$ for which $P(D)u \in L^2(\Omega)$ in the distribution sense: (ii) The very strong extension P_S which is the closure of $P(D)$ defined for those u which are restrictions to Ω of functions in $C_0^\infty(\Omega)$, the set of infinitely differentiable functions with compact support in R^ν.

In the present paper it is proved that: (1) If Ω satisfies a smoothness condition we always have $P_\omega = P_S$. (2) If $\nu > 1$, we can always find an open set for which $P_\omega \neq P_S$. Moreover, the first statement holds for operators with smooth variable coefficients which are formally hypoelliptic in the sense of the author [#5064]. The method of proof also solves another problem posed by the author [(*), #5064]. *M. Schechter* (New York, N.Y.)

5068:

Schwarz, Stephan. On weak and strong extensions of partial differential operators with constant coefficients. Ark. Mat. **3** (1958), 515–526.

We retain the notation and definitions of the review above. If $C^\infty(\Omega)$ denotes the set of infinitely differentiable functions defined in Ω, the closure of $P(D)$ defined for all $u \in C^\infty(\Omega)$ such that u and $P(D)u$ are in $L^2(\Omega)$ is called the strong extension of $P(D)$ and is denoted by P_s. (Compare this with the definition of the very strong extension given above.) Hörmander [(*), #5064] proved that $P_s = P_\omega$ when $P(D)$ is of local type and Ω is any domain. The present author proves that if $P(D)$ is homogeneous and not of local type, one can find an open set Ω for which $P_s \neq P_\omega$. He does this by modifying an unpublished example given by Hörmander for $P(D) = \partial^2/\partial x \partial y$.

M. Schechter (New York, N.Y.)

5069:

Wolska-Bochenek, J. Un problème aux limites à dérivée tangentielle pour l'équation du type elliptique. Ann. Polon. Math. **4** (1958), 275–287.

The following boundary value problem is treated in two dimensions:

$$(1) \qquad \Delta u = F(x, y, u, u_x, u_y)$$

for (x, y) in the interior of a domain D bounded by a closed curve C on which

$$(2) \qquad \frac{du}{dn} + a(s)u = \Phi\left(s, u, \frac{du}{ds}\right),$$

du/dn and du/ds being the normal and tangential derivatives of u; a, F, and Φ are given functions. Under certain conditions on a, F, Φ, and C it is shown that if the problem with $F \equiv \Phi \equiv 0$ admits only the solution $u \equiv 0$, then there exists at least one solution of (1) and (2).

J. Elliott (New York, N.Y.)

5070:

Morrey, Charles B., Jr. On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. I. Analyticity in the interior. Amer. J. Math. **80** (1958), 198–218.

Elliptic systems of analytic, non-linear partial differential equations of the type introduced by Douglis and Nirenberg [Comm. Pure Appl. Math. 8 (1955), 503–538; MR 17, 743] are the systems discussed. It is shown that any solution of such a system possessing a sufficient number of Hölder-continuous derivatives is analytic at each interior point of its domain. The proof begins by changing the dependent variables to new variables $v = (v^k)$, which vanish to certain orders at an arbitrarily selected interior point taken as the origin 0. As a result of the change, the system of differential equations can be written as

$$(*) \qquad Lv = Mv + \psi(x, v, \cdots),$$

where $Lv = \{\sum_k L_{jk} v^k\}$ is an elliptic system of linear differential expressions with constant coefficients, each differential operator L_{jk} being of homogeneous order $(= s_j + t_k)$, $Mv = \{\sum_k M_{jk} v^k\}$ is a system of linear differential expressions with constant coefficients such that the order of M_{jk} is less than that of L_{jk}, and $\psi = \{\psi_j\}$ is a system of functions of the independent and dependent variables and of the derivatives of the latter up to the orders occurring in L. In the Taylor expansions of the ψ_j (with respect to all arguments) about 0, only terms of second and higher degree occur, except possibly for first degree terms in the independent variables x. Within a ball $B_R : |x| < R$, the system (*) is reduced to a certain other type of functional relation. The reduction depends on the use of appropriate generalized potentials to associate to each vector f of a certain Hölder continuity class in B_R a vector $u = P_R(f)$, belonging to another such class, for which $Lu = f$. F denoting the right member of (*), let $V_R = P_R(F)$ and $v = V_R + H_R$. The functional relation referred to, which is equivalent to (*), can be written in terms of the function $W_R = P_R(MH_R + \psi(x, H_R, \cdots))$ and the transformation (of V_R)

$$T_R(V_R; H_R) = P_R\{MV_R + \psi(x, H_R + V_R, \cdots) - \psi(x, H_R, \cdots)\}$$

as

$$(**) \qquad V_R - T_R(V_R; H_R) = W_R.$$

Because of the indicated properties of P_R and ψ, if R is sufficiently small, V_R can be found in B_R from (**) by iterations, if only H_R is known (v being known, actually both H_R and V_R are known). Furthermore, if the function H_R can be extended analytically to a complex domain, if the transformation T_R can be extended analytically to act on complex analytic functions, and if the extended T_R is subject to inequalities of the same type as had controlled previously the iterative process in the real case, a similar iterative process will succeed in the complex case in producing a complex analytic V_R over a suitable complex domain, which satisfies the extended version of (**). Thereby, $v = V_R + H_R$ will be seen to be analytic, as contended. The remaining steps in the proof of the analyticity of v thus are concerned with analytic continuation into the complex. The continuability of H_R depends on the fact that this function satisfies an elliptic system, namely $LH_R = 0$, of homogeneous, linear equations with constant coefficients. The continuability of T_R depends on the fact that the generalized potentials constructed can be extended to complex domains, this being accomplished by a device used by E. E. Levi [Rend. Circ. Mat. Palermo **24** (1907), 275–317] and E. Hopf [Math. Z. **34** (1932), 194–233]. Analyticity for systems of linear and non-linear equations previously have been discussed by Petrowsky [Mat. Sb. (N. S.) **5** (**47**) (1939), 3–70; MR **1**, 236], Morrey

and Nirenberg [Comm. Pure Appl. Math. **10** (1957), 271–290; MR **19**, 654], and Avner Friedman [J. Math. Mech., to appear]. *A. Douglis* (College Park, Md.)

5071:

Baranovskiĭ, F. T. The Cauchy problem for a linear second-order hyperbolic differential equation degenerating on the initial plane. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 227–254. (Russian)

This paper concerns a linear partial differential equation of the form

$$\varphi(t)\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + f.$$

Letting $k=[3n/2]+5$, it is assumed that

$$\varphi(t),\ a_{ij}(x_1, \cdots, x_n) \in C^{k+1};\quad b_i, c, f \in C^k;$$

and that, for certain positive constants $c_1, \cdots, c_4, \alpha$, $c_1 t^\alpha \leqq \varphi(t) \leqq c_2 t^\alpha$, $c_4 t^{\alpha-1} \leqq \varphi'(t) \leqq c_3 t^{\alpha-1}$, where $0 \leqq \alpha < 1$. It is also assumed that the matrix (a_{ij}) is positive definite. It is shown that there exists one and only one solution of such an equation satisfying initial conditions of the form

$$u\Big|_{t=0} = \psi_0(x_1, \cdots, x_n),\quad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi_1(x_1, \cdots, x_n),$$

where ψ_0 and ψ_1 are assumed $\in C^{k+2}$. The solution $\in C^2$ in $x_1, \cdots, x_n$ for $t \geqq 0$ and it is also $\in C^2$ in t for $t>0$, while the product $\varphi(t)u_{tt}$ is continuous at $t=0$.

D. C. Lewis, Jr. (Baltimore, Md.)

5072:

★**Bureau, F. J. Le problème de Cauchy pour une classe d'équations linéaires totalement hyperboliques du second ordre.** Symposium on the numerical treatment of partial differential equations with real characteristics: Proceedings of the Rome Symposium (28–29–30 January 1959) organized by the Provisional International Computation Centre, pp. 131–135. Libreria Eredi Virgilio Veschi, Rome, 1959. xii+158 pp.

The Cauchy problem for

$$u_{tt} = a^{ij}(x)u_{x^i x^j} + b^i(x)u_{x_i} + c(x),$$

a^{ij} positive definite, coefficients analytic, is solved fairly explicitly, following Hadamard.

P. Ungar (New York, N.Y.)

5073:

Lewis, Robert M. Discontinuous initial value problems for symmetric hyperbolic linear differential equations. I. J. Math. Mech. **7** (1958), 571–592.

This paper extends the work of R. Courant and P. D. Lax [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 872–876; MR **18**, 399] by considering (a) characteristics of constant multiplicity higher than 1 and (b) mixed initial and boundary value problems involving reflection of discontinuities. By means of this extension, certain problems involving Maxwell's equations can be treated. (See the following review.) *D. Ludwig* (New York, N.Y.)

5074:

Lewis, Robert M. Asymptotic expansion of steady-state solutions of symmetric hyperbolic linear differential equations. II. J. Math. Mech. **7** (1958), 593–628.

The author generalizes results of R. K. Luneberg to apply to a wide class of symmetric hyperbolic equations [see preceding review]. By exploiting the connection between geometrical optics and discontinuous initial-value problems, many of the techniques of geometrical optics can be rigorously justified. Certain cases remain outside the scope of the theory, most important of which are problems involving shadows (diffraction) and equations whose characteristics have variable multiplicity.

D. Ludwig (New York, N.Y.)

5075:

Lewis, Robert M. Discontinuous initial value problems and asymptotic expansion of steady-state solutions. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. MME-8 (1957), 85 pp.

This report consists of the two papers reviewed above.

D. Ludwig (New York, N.Y.)

5076:

Kahane, Arno. Sur quelques transformations du type Bäcklünd relatives à des équations de la forme $z_{uv}=f(z)$. Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 415–438. (Romanian. Russian and French summaries)

Die Differentialgleichungen des genannten Typs haben die Mathematiker bereits viel beschäftigt. Man bemühte sich meist darum, zu einer gegebenen Lösung mit Hilfe von Transformationen, z. B. nach Bäcklund, weitere Lösungen zu finden oder diese Lösungen aus integrierbaren Hilfssystemen erster Ordnung zu erhalten. Verf. gibt hierfür vereinfachte Beweise und bringt viele weitere Einzelheiten, die aber nicht immer neu sind. Seine Betrachtungen betreffen hauptsächlich die Spezialfälle $z_{uv}=\sin z$ und $z_{uv}=z$. *W. Burau* (Hamburg)

5077:

Gehring, F. W. On solutions of the equation of heat conduction. Michigan Math. J. **5** (1958), 191–202.

L'auteur reprend dans le plan l'étude des fonctions paraboliques (intégrales de $\partial^2 u/\partial x^2 = \partial u/\partial t$) ou sous-paraboliques, comparée à celle des fonctions harmoniques ou sousharmoniques. [Mais sa bibliographie oublie en particulier Doob, Trans. Amer. Math. Soc. **80** (1955), 216–280; MR **18**, 176.] Il s'occupe essentiellement de w sous-parabolique dans une bande $0<t<c$. Soit $k(x,t)=(1/4\pi)t^{1/2}\exp(-x^2/4t)$; la condition

$$\int_{-\infty}^{+\infty} dx \int_0^a k(x, b-t)w^+ dt < \infty$$

pour tous les $0<a<b<c$ entraîne que w est majorée par la borne supérieure des lim sup aux points de $t=0$. La condition

$$\sup_{0<t<b} \int_{-\infty}^{+\infty} k(x, b-t)w^+ dx < \infty$$

pour tous les $0<b<c$ équivaut à l'existence d'une majorante parabolique. La condition analogue pour w parabolique et $|w|$ au lieu de w^+ équivaut à la possibilité de la représentation intégrale $\int_{-\infty}^{+\infty} k(x-y, t)d\mu(y)$.

M. Brelot (Paris)

5078:

Pogorzelski, W. Étude d'une fonction de Green et du

problème aux limites pour l'équation parabolique normale. Ann. Polon. Math. **4** (1958), 288–307.

The Green's function is determined for a parabolic equation

$$(1)\quad \Psi(u) = \sum_{i,j=1}^{n} a_{i,j}(A,t)\partial^2 u/\partial x_i \partial x_j + \sum_{i=1}^{n} b_i(A,t)\partial u/\partial x_i + c(A,t)u - \partial u/\partial t = 0,$$

where A lies in a closed, bounded region R of n-dimensional euclidean space ($n \geq 2$) bounded by the surface S. The coefficients are assumed to satisfy Hölder conditions. The properties of the Green's function are studied and application is made to the solution of the inhomogeneous equation $\Psi(u) = F$ under the conditions

$$(2)\quad \lim_{A\to P} u(A,t) = 0, \quad \lim_{t\to 0} u(A,t) = 0,$$

where A is an interior point of R and P lies on S.

J. Elliott (New York, N.Y.)

5079:

Piskorek, A. Sur certains problèmes aux limites pour l'équation semi-linéaire parabolique normale. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 505–510.

This communication is concerned with the existence of solutions of a mixed boundary value problem for semi-linear parabolic equations

$$H[u(X,t)] \equiv \sum a_{ij}(X,t)\partial^2 u/\partial x_i \partial x_j + \sum b_k(X,t)\partial u/\partial x_k + c(X,t)u - \partial u/\partial t = F(X,t,\partial u/\partial x_1,\cdots,\partial u/\partial x_n).$$

$X = (x_1,\cdots,x_n)$ is restricted to a bounded domain in $E^{(n)}$ with Liapounoff boundary S. The boundary conditions have the form $\lim u(X,t) = f(X)$ as $t\to 0$, and $\partial u/\partial \nu = G(P,t,u(P,t))$, where $\partial u/\partial \nu$ is the normal derivative at $P\in S$ and $0 < t \leq T$. The first problem discussed treats the case $f(X) \equiv 0$. The functions a_{ij}, b_k, c, F and G are required to satisfy uniform Hölder conditions in all indicated variables except t, and for the a_{ij} the Hölder condition includes t. $G(P,t,v)$ is assumed continuous down to $t=0$ with $G(P,0,v)=0$ for $P\in S$, $|v| \leq R$. Based on properties of the fundamental solution for the homogeneous equation $H[u]=0$, obtained by W. Pogorzelski [Ricerche Mat. **5** (1956), 25–57; **6** (1957), 162–194; MR **18**, 47; **20** #1845; and #5078 above], the author reduces the problem to the solution of a set of integro-differential equations. Assuming certain relations between the bounds $|G|$, $|F|$, R and the Hölder exponents, the existence of a solution is guaranteed by the Schauder fixed point theorem. For the case $f(X) \not\equiv 0$ additional postulates involving t in the Hölder continuity are required, but the methods are not essentially different.

A. N. Milgram (Berkeley, Calif.)

5080:

Pogorzelski, W. Premier problème de Fourier pour l'équation parabolique dont les coefficients dépendent de la fonction inconnue. Ann. Polon. Math. **6** (1959/60), 15–40.

The author proves the existence of a solution of the first mixed boundary problem for the parabolic equation

$$\sum a_{ij}\partial^2 u/\partial x_i \partial x_j + \sum b_i \partial u/\partial x_i + cu - \partial u/\partial t = F$$

in a cylindrical domain. The coefficients a_{ij}, b_i, c are assumed to be Hölder continuous in (x,t,u) and F is assumed to be Hölder continuous in $(x,t,u,\partial u/\partial x_i)$. The Hölder coefficients of the above functions with respect to u and $\partial u/\partial x_i$ are restricted to be very small. The proof uses Green's function [constructed by the author in #5078 above] to transfer the given problem into a problem of solving an integro-differential equation. The latter is then treated by a fixed-point-theorem technique.

A. Friedman (Minneapolis, Minn.)

5081:

Mizohata, Sigeru. Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **31** (1958), 219–239.

The author presents in this paper the following fundamental result, concerning uniqueness of the continuation of a solution of a parabolic equation

$$(1)\quad \frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u,$$

the coefficients of which are supposed to be real and infinitely differentiable, while

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \delta(x,t)|\xi|^2, \quad \delta(x,t) > 0.$$

If $u(x,t)$ is a solution of (1), defined in a neighbourhood of the origin and satisfying the conditions (2) $u(x,t)=0$ and $\partial u/\partial x_n = 0$ for $x_n = 0$, then $u(x,t) \equiv 0$ in a neighbourhood of the origin. The hyperplane $x_n = 0$ may be replaced by an infinitely differentiable hypersurface with non-horizontal tangent plane in the origin. The second condition (2) is then to be replaced by $\partial u/\partial n = 0$.

The proof of the theorem is based on Calderón's and Zygmund's method, used by Calderón in Amer. J. Math. **80** (1958), 16–36 [MR **21** #3675]. This method is based on a representation of linear differential operators by means of singular integral operators: If A is a linear differential operator of homogeneous order m defined on all Euclidean space and with bounded coefficients, then $Au = H\Lambda^m u$, where Λ is a square root of the negative of the Laplacian and H is a singular integral operator. Substantial in Calderón's work is that the differential operator A has no multiple characteristics. The author arrives at his result for parabolic equations by generalising Calderón's operators and by extending their fundamental properties to the generalised ones.

K. Rektorys (Prague)

5082:

Milicer Gružewska, H. Propriété limite du potentiel généralisé de simple couche relativement à l'équation parabolique normale. Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 359–396.

The generalized simple layer potential corresponding to the parabolic equation

$$(1)\quad \sum_{i,j=1}^{n} a_{ij}(A,t)\partial^2 u/\partial x_i \partial x_j + \sum_{i=1}^{n} b_i(A,t)\partial u/\partial x_i + C(A,t)u = \partial u/\partial t$$

with fundamental solution Γ, is the surface integral

$$(2)\quad U(A,t) = \int_0^t \iint_S \Gamma(A,t;Q,\tau)\varphi(Q,\tau)\,dQ\,d\tau,$$

where A lies in an n-dimensional domain Ω ($n \geqq 3$) bounded by a closed surface S, and φ is a given density. In this paper, the behavior of $U(A, t)$ as $t \to \infty$ is studied. It is shown, under rather general conditions on Ω, φ, and the coefficients of (1), that $U(A, t)$ tends to the simple layer potential of the limiting elliptic equation

$$(3) \quad \sum_{i,j=1}^{n} \bar{a}_{ij}(A)\partial^2 u/\partial x_i \partial x_j + \sum_{i=1}^{n} \bar{b}_i(A)\partial u/\partial x_i + \bar{c}(A)u = 0,$$

whose coefficients are the limits as $t \to \infty$ of those in (1). It is assumed that these limits, as well as that of $\varphi(Q, t)$, exist uniformly, and that the coefficients of (1) and (3) satisfy certain Hölder conditions. The precise formulation of all the conditions is too lengthy to present here.

J. Elliott (New York, N.Y.)

5083:

★**De Giorgi, E. Considerazioni sul problema di Cauchy per equazioni differenziali di tipo parabolico di ordine qualunque.** Symposium on the numerical treatment of partial differential equations with real characteristics: Proceedings of the Rome Symposium (28–29–30 January 1959) organized by the Provisional International Computation Centre, pp. 136–139. Libreria Eredi Virgilio Veschi, Rome, 1959. xii + 158 pp.

This is a report of results of the author [Rend. Mat. e Appl. (5) **14** (1955), 382–387; Ann. Mat. Pura Appl. (4) **40** (1955), 371–377; MR **16**, 1119; **17**, 748] and on several related open problems.

L. Nirenberg (New York, N.Y.)

5084:

Slobodeckiĭ, L. N. Generalized solutions of parabolic and elliptic systems. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 809–834. (Russian)

Soit $N^{(q)}$ [resp. $S^{(q)}$] l'espace des fonctions $t \to f(t)$, $t \in (0, T)$, qui sont L^1 [resp. L^∞] à valeurs dans $L^q(\Omega)$, $q \geqq 1$, Ω ouvert de R^n. Si f est donnée sommable sur $\Omega \times (0, T)$, on considère

$$u_\lambda(t, x) = \int_0^t ds \int_\Omega \exp\left(-\eta|x-y|^{2p/(2p-1)}(t-s)^{-1/(2p-1)}\right) \times (t-s)^{-(n+\lambda)/2p} f(s, y)dy,$$

$0 \leqq \lambda < 2p$ (potentiels paraboliques). L'A. établit quelques résultats du type de ceux de Sobolev pour les potentiels ordinaires. Par exemple, si $q \leqq n(2p-\lambda)^{-1}$ et si $f \in S^{(q)}$ alors $u \in S^{(q_1)}$, où $q \leqq q_1 < nq(n-q(2p-\lambda))^{-1}$. Applications intéressantes aux systèmes paraboliques.

J. L. Lions (Nancy)

5085:

Ovsyannikov, L. V. Group relations of the equation of non-linear heat conductivity. Dokl. Akad. Nauk SSSR **125** (1959), 592–495. (Russian)

The author investigates the group properties of the non-linear equation

$$\frac{\partial}{\partial x}\left(f(u)\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial t}$$

and finds the basic group for arbitrary $f(u)$. $f(u)$ is also specialized to each of the forms e^u, u^{2m} ($m \neq -2/3$), $u^{-4/3}$. Solutions are given for the various cases.

C. G. Maple (Ames, Iowa)

5086:

Nižnik, L. P. On the spectrum of general differential operators. Dokl. Akad. Nauk SSSR **124** (1959), 517–519. (Russian)

Let $P(\xi)$ [$\xi = (\xi_1, \cdots, \xi_m)$] be a complete polynomial and let $Q_j(\xi)(j=1, 2, \cdots, k)$ be another polynomial such that

$$|Q_j(\xi)|[|P(\xi)|+1]^{-1} \leqq C < \infty,$$

$$[\sum_\alpha |Q_j^{(\alpha)}(\xi)|^2]^{1/2}[\sum_\alpha |P^{(\alpha)}(\xi)|^2]^{-1/2} \leqq C(1+|\xi|)^{-1}$$

for all $\xi \in R^m$. Let $c_j(x)$ ($j=1, 2, \cdots, k$; $x \in R^m$) be a (g_j+2m+1)-times differentiable function (g_j the degree of Q_j), with the last derivatives integrable on R^m. Suppose that

$$P(D) + \sum_{j=1}^{k} c_j(x)Q_j(D), \quad \left[D = \left(\frac{1}{i}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_m}\right)\right]$$

is formally selfadjoint. The author's main result is: The smallest closed operator [in $L^2(R^m)$] containing the preceding operator defined on the space $\mathscr{D}$ of L. Schwartz is selfadjoint, and its limit spectrum coincides with the spectrum of the selfadjoint operator generated [in $L^2(R^m)$] by $P(D)$ defined on $\mathscr{D}$.

C. Foiaş (Bucharest)

5087:

★**Lions, J. L. Quelques applications d'opérateurs de transmutation.** La théorie des équations aux dérivées partielles. Nancy, 9–15 Avril 1956, pp. 125–137. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXI. Centre National de la Recherche Scientifique, Paris, 1956. 187 pp. 1500 francs.

Soit $L = D^2 - q(x)$, $D = d/dx$, $q(x)$ étant une fonction indéfiniment différentiable à valeurs réelles ou complexes. $\mathscr{H}$ est un espace de fonctions ou de distributions sur R. Un opérateur linéaire continu X de $\mathscr{H}$ dans lui-même est dit opérateur de transmutation (de L en D^2), si $D^2XT = XLT$ pour tout $T \in \mathscr{H}$ et si en outre X est un isomorphisme de $\mathscr{H}$ sur lui-même. L'A. montre que, dans plusieurs espaces $\mathscr{H}$, il existe des transmutations de L en D^2. Prenons un cas: $\mathscr{H} = \mathscr{D}_+'$ (ou $\mathscr{D}_+$), espace des distributions (ou des fonctions indéfiniment différentiables) à support limité à gauche. Alors

$$Xf(x) = f(x) + \int_{-\infty}^{x} X(x, y)f(y)dy$$

définit une transmutation, $X(x, y)$ étant la solution d'un problème aux limites relatif à $(\partial^2/\partial x^2 - \partial^2/\partial y^2 + q(y))X = 0$.

Ceci établi, l'A. montre que, pour certains opérateurs, les problèmes mixtes se traitent d'une manière simple en utilisant des opérateurs de transmutation. Il souligne aussi que l'opérateur de Green se calcule par un procédé explicite. Deuxièmement, les problèmes mixtes singuliers se traitent du même point de vue. On trouve les démonstrations détaillées dans l'article de l'auteur, Bull. Soc. Math. France **84** (1956), 9–95 [MR **19**, 556].

S. Mizohata (New York, N.Y.)

5088:

Zerner, Martin. Solution élémentaire locale d'équations aux dérivées partielles dépendant d'un paramètre. C. R. Acad. Sci. Paris **248** (1959), 3679–3681.

Soit $P(y, D_x)$ un opérateur différentiel à coefficients constants dans un ouvert Ω de R_x^n, l'origine $O \in \Omega$, dépendant analytiquement du paramètre $y \in$ variété

analytique Y. On suppose que, pour tout $y_1, y_2 \in Y$, $P(y_1, D_x)$ et $P(y_2, D_x)$ sont équivalents au sens de Hörmander [Acta Math. **194** (1955), 161–248; MR **17**, 853]. Soit ω un voisinage ouvert de O, relativement compact dans Ω. Alors, pour tout $y_0 \in Y$, il existe un voisinage V de y_0 et une distribution $F_{(y)}$, fonction analytique de $y \in V$ à valeurs dans l'espace des distributions à support compact dans Ω et d'ordre $\leqq [\frac{1}{2}n]+1$ en L^2, avec

$$P(y, D_x)F_{(y)} = \delta + S_{(y)},$$

$S_{(y)}$ distribution dont le support demeure dans $\complement\omega$.

J. L. Lions (Nancy)

5089:

Bakel'man, I. Ya. Regularity of solutions of Monge-Ampère equations. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 143–184. (Russian)

The Dirichlet problem for the Monge-Ampère equation (1) $rt-s^2=\varphi(x, y, z, p, q)$ in the circle $x^2+y^2 \leqq r^2$ was studied by S. Bernstein [Soobšč. Har'kov. Mat. Obšč. **11** (1907), 1–164] who obtained estimates for the bounds of the solutions $z(x, y)$ under certain analyticity and boundedness conditions. The author extends this method to find new estimates for $|z(x, y)|$ and its first and second derivatives (e.g. $|z(x, y)| \leqq (2-\sqrt{3})r+r^2+3 \max_{0 \leqq \theta \leqq 2\pi} |\psi(\theta)|$, where $\psi(\theta)$ is the boundary function for the Dirichlet problem and r is further restricted). He then studies the Dirichlet problem of (1) with $\varphi(x, y, z, p, q)=A(x, y)R(p, q)$ considering the generalized solutions as defined in his earlier paper [Dokl. Akad. Nauk SSSR **114** (1957), 1143–1145; MR **20** #1983] and shows that there are two such solutions, both convex and at least three times differentiable [cf. A. D. Aleksandrov, *Vypuklye mnogogranniki*, Gosudarstv. Izdat. Tehn. Teor. Lit., Moscow-Leningrad, 1950; MR **12**, 732].

V. Linis (Ottawa, Ont.)

5090:

Drăgan, I. Sur les intégrales premières des systèmes des caractéristiques pour les équations aux dérivées partielles du troisième ordre à caractéristiques distinctes. Acad. R. P. Romîne. Fil. Iași Stud. Cerc. Ști. Mat. **9** (1958), no. 2, 141–162. (Romanian. Russian and French summaries)

The author studies a non-linear partial differential equation of the third order in two independent variables under the assumption that the characteristics of the equation are distinct. He denotes by (C_i^n), $i=1, 2, 3$, the three systems of characteristic differential equations of order n, and studies first integrals of these systems. His results for $n>3$ are as follows. I. There is, in addition to the integrals of (C_i^{n-1}), at most one integral of order n of (C_i^n). II–IV. For $p=1, 2, 3$, there are at most p integrals of order p of (C_i^n). V, VI. For $p=2, 3$, the number of integrals of order p cannot exceed the number of integrals of orders $p-1$, $p-2$.

A. Erdélyi (Pasadena, Calif.)

5091:

★Helgason, Sigurdur. Partial differential equations on Lie groups. Treizième congrès des mathématiciens scandinaves, tenu à Helsinki 18–23 août 1957, pp. 110–115. Mercators Tryckeri, Helsinki, 1958. 209 pp. (1 plate)

The mean-value theorem of Ásgeirsson is the following. Let $u=u(x_1, \cdots, x_n, y_1, \cdots, y_n)$ be a C^2-function on $R^n \times R^n$. If u satisfies $\Delta_x u=\Delta_y u$, then

$$\int_{S^r(x^0)} u(x, y^0)d\omega(x) = \int_{S^r(y^0)} u(x^0, y)d\omega(y),$$

where Δ_x, Δ_y denote the Laplace operators with respect to x, y, x^0 and y^0 are fixed points in R^n, and where $d\omega(x)$ denotes the euclidean surface element on the sphere $S^r(x^0)$ of radius r with center x^0. This paper mainly states, without detailed proof, the following two theorems. (1) The above theorem can be generalized to the case where R^n is replaced by a homogeneous space with invariant Riemannian metric, Δ by invariant analytic linear differential operators, S^r by geodesic spheres, $d\omega$ by Haar measure, and where u is restricted to analytic functions. (2) If we further restrict the space, Ásgeirsson's theorem can be generalized for C^2-functions u. The restriction in (2) is that (i) the group of isotropy is transitive on directions, and (ii) the homogeneous space is simply connected.

M. Kuranishi (Nagoya)

POTENTIAL THEORY

See also 5057, 5077, 5082, 5084, 5240, 5241, 5242.

5092:

Gagua, M. B. On completeness of systems of harmonic functions. Soobšč. Akad. Nauk Gruzin. SSR. **19** (1957), no. 1, 3–10. (Russian)

Let G be a bounded region in 3-space and $\mathfrak{M}$ the Hilbert space of all bounded harmonic functions on G under the L^2 norm. Further, let the complement of $\bar{G}$ have infinitely many components $G_i'(i=0, 1, \cdots)$, where G_0' is the unbounded component. (For consistency and clarity the notation here differs from that of the author.) Sequences $\{V_k(P)\}_{k=0}^\infty$ and $\{V_k^*(P)\}_{k=0}^\infty$, respectively, are obtained by orthonormalizing over G the functions

$$r_{p_i}^{-n-1}P_n^m(\cos\vartheta_{p_i}) \begin{matrix}\cos m\varphi_{p_i} \\ \sin m\varphi_{p_i}\end{matrix} \quad (m \leqq n;\, n, i = 0, 1, \cdots)$$

and

$$r^n P_n^m(\cos\vartheta) \begin{matrix}\cos m\varphi \\ \sin m\varphi\end{matrix} \quad (m \leqq n;\, n = 0, 1, \cdots),$$

where $\{p_i\}$ is a fixed sequence of points $p_i \in G_i'$; r_{p_i}, ϑ_{p_i}, φ_{p_i} are spherical coordinates of a point P measured from p_i; r, ϑ, φ are spherical coordinates of P measured from the origin; and $P_n^m(x)$ are the associated Legendre functions.

Theorem: Let G satisfy the conditions (a) $\partial G = \bigcup_{i=0}^\infty \partial G_i'$ (∂E denoting the boundary of E), and (b) every regular solution F on G of $\Delta^2 F=0$ (Δ denoting the Laplacian) for which $F=F_x=F_y=F_z=0$ on ∂G vanishes identically. Then every $u \in \mathfrak{M}$ admits a Fourier expansion in the functions V_k, the convergence being uniform on compact sets. If G is simply connected, then u also admits a corresponding expansion in the functions V_k^*. [The biharmonic condition (b) enters the proof via the fact that potentials F having density u are used, so that $\Delta^2 F = -4\pi\Delta u=0$.] This theorem leads to results on the completeness (relative to uniform convergence) of $\{V_k\}$ and $\{V_k^*\}$ over certain subspaces of $\mathfrak{M}$, generalizing theorems of Keldych-Lavrientieff and Vekua.

M. G. Arsove (Seattle, Wash.)

5093:

★**Séminaire de théorie du potentiel, dirigé par Marcel Brelot et Gustave Choquet; 1re année: 1957.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958. iii+61 pp. (mimeographed)

In the first exposition G. Choquet proves many of the results announced in two notes [C.R. Acad. Sci. Paris **243** (1956), 635–638; **244** (1957), 1606–1609; MR **18**, 295; **19**, 405] and some supplementary propositions. In the second and fourth, L. Naïm presents most of her paper [Ann. Inst. Fourier, Grenoble **7** (1957), 183–281; MR **20** #6608], including proofs. In the fifth, J. Deny proves many of the results announced in his note with A. Beurling [#5098 below]. In the sixth, M. Brelot develops a theory of generalized subharmonic functions and Dirichlet problems on a locally compact space. The axioms and their consequences are presented more definitively in two later notes [#5094a–b below]. The third exposition, by J. L. Doob, appears only in résumé and the seventh exposition, by G. Choquet, appears only in title.

G. A. Hunt (Princeton, N.J.)

5094a:

Brelot, Marcel. Extension axiomatique des fonctions sous-harmoniques. C. R. Acad. Sci. Paris **245** (1957), 1688–1690.

5094b:

Brelot, Marcel. Extension axiomatique des fonctions sous-harmoniques. C. R. Acad. Sci. Paris **246** (1958), 2334–2337.

The notes present a nearly definitive axiomatic treatment of an abstract situation generalizing the usual setting of the Dirichlet problem and subharmonic functions.

Let Ω be a locally compact space, connected and not compact. To each open set ω in Ω is assigned a real vector space of continuous functions on ω, the principal functions. An open set ω is said to be regular if it has compact closure, if every continuous function f on its boundary ω^* can be extended continuously to $\omega \cup \omega^*$ by a unique principal function H_f, and if H_f increases with f. The first note assumes the following axioms to hold. (I) If u is principal on ω and $\omega_0 \subset \omega$, then the restriction of u to ω_0 is principal; if $\omega = \bigcup \omega_i$, then u is principal on ω if its restriction to each ω_i is principal. (II) The connected regular sets form a basis for the topology. (III) Every increasing directed system of principal functions on a connected open set tends either to $+\infty$ or to a principal function.

Solutions of a linear second order elliptic equation form a typical example of a system of principal functions; the third axiom unfortunately rules out solutions of a parabolic equation.

Principal functions, or p-functions, are called harmonic functions in some later notes. The p-function H_f above can be written $H_f(x) = \int f d\rho_\omega^x$, with ρ_ω^x a finite positive measure on ω^*. These measures play the role of harmonic measures, and one defines $\bar{p}^*$-functions in terms of them as the strict analogues of ordinary super-harmonic functions; the hyperprincipal or $\bar{p}$-functions are defined similarly, but are permitted to be $+\infty$ on open sets. The $\bar{p}^*$-functions enjoy many of the properties of superharmonic functions. In addition, the Perron-Wiener-Brelot method of stating and solving the Dirichlet problem carries over, the boundary being any sufficiently large class of filters on Ω, none of which has a point of adherence in Ω itself.

In the second note the author first studies the class of functions obtained by dropping the condition of lower semicontinuity in defining $\bar{p}$-functions. The results are used in establishing, under additional axioms, the Riesz-Martin representation of $\bar{p}$-functions: Fix a regular set ω_0 and one of its points x_0, and denote by $\mathcal{E}^+_{x_0,\omega_0}$ the set of nonnegative $\bar{p}$-functions v on Ω satisfying $\int v d\rho_{\omega_0}^{x_0} = 1$. Every function v in $\mathcal{E}^+_{x_0,\omega_0}$ is the sum of a nonnegative principal function and a potential (that is, a $\bar{p}^*$-function having zero for greatest minorant among the $\bar{p}$-functions), and moreover v is a unique mean of extreme elements of $\mathcal{E}^+_{x_0,\omega_0}$. In proving these results the author assumes Ω to be separable and uses the following axioms. (III′) For each domain δ and each point x_0 of δ, the positive $\bar{p}$-functions on δ with value 1 at x_0 are equicontinuous at x_0. (IV) The sets which are regular and determining form a basis for the topology of Ω. (A regular set ω is determining if every function which is nonnegative, locally bounded, hyperprincipal on Ω and principal on ω, is determined by its values outside ω.)

The proofs are not given, but most of them can be reconstructed from the exposition in *Séminaire Brelot-Choquet*, 1957 [#5093 above]. The 1958 Séminaire develops in detail the notes reviewed here and in #5095 and #5097.

G. A. Hunt (Princeton, N.J.)

5095:

Brelot, Marcel. La convergence des fonctions surharmoniques et des potentiels généralisés. C. R. Acad. Sci. Paris **246** (1958), 2709–2712.

In two earlier papers [#5094a–b above] the author has given an axiomatic discussion of harmonic and subharmonic functions. In the present paper he continues this discussion. Here the three axioms of the first paper are supplemented by a fourth axiom which requires that every potential which dominates a locally bounded potential V on a certain set dominates it everywhere. A set E contained in an open subset Ω_0 of Ω is defined as polar in Ω_0 if there exists a non-negative $\bar{p}^*$ function equal to $+\infty$ at least on E. A statement is said to be valid "quasi-partout" if it is valid except on a polar set. Several theorems are given, one of which asserts that the lower envelope of a family of non-negative $\bar{p}$ functions is a certain p function "quasi-partout". In particular, consider a set E and a nonnegative $\bar{p}^*$-function v; denote by R_v^E the lower envelope of the nonnegative $\bar{p}$-functions that dominate v on E; on replacing $R_v^E(x)$ by $\liminf_{y\to x} R_v^E(y)$, one obtains a $\bar{p}^*$-function $\hat{R}_v^E$, the extremisation of v for the complement of E; now, by the theorem mentioned, $\hat{R}_v^E$ is the least nonnegative $\bar{p}$-function that dominates v "quasi-partout" on E.

F. W. Perkins (Hanover, N.H.)

5096:

Hervé, Rose-Marie; et Brelot, Marcel. Introduction de l'effilement dans une théorie axiomatique du potentiel. C. R. Acad. Sci. Paris **247** (1958), 1956–1959.

We use the same definitions and notations as in the preceding three notes [reviews above] and in Exp. 6 of #5093. We call functions previously named p-, $\bar{p}$- and $\bar{p}^*$-functions harmonic, superharmonic in the wide sense

and superharmonic, respectively. Let Ω be a locally compact, non-compact, Hausdorff space with a countable base, and assume axioms 1, 2, 3, 4′ and the existence of a positive potential. A set E is called thin at $x_0 \notin E$ if $x_0 \notin \bar{E}$ (closure) or a positive superharmonic function $v(x)$ exists in Ω such that $\liminf v(x) > v(x_0)$ as $E \ni x \to x_0$. If $x_0 \in E$, E is called thin at x_0 whenever x_0 is polar and $E - x_0$ is thin at x_0. The present paper is concerned with thin sets. First a necessary and sufficient condition for E to be thin at $x_0 \notin E$ is given and it is proved that, given a set E, the points of E where E is thin form a polar set. Next it is shown that a closed set E is not thin at a boundary point x_0 of E if and only if there exists a positive superharmonic function outside E which tends to zero as $x \to x_0$. As an application, it follows that, for a locally bounded superharmonic function u with support S in an open set ω with $\bar{\omega} \subset \Omega$, $\limsup u(x)$ taken as $S \not\ni x \to x_0 \in S^*$ is equal to $\limsup u(x)$ taken as $S^* \ni x \to x_0$, where S^* is the boundary of S. Consequently, if the restriction of u to S is continuous at x_0, then u is continuous at x_0 as a function in ω. As in the classical Dirichlet problem, a (finite-valued) continuous boundary function is resolutive ($\underline{H}_f^{\omega} = \overline{H}_f^{\omega}$) and the solution takes the boundary value at points where the complement of ω is not thin. It is proved also that the boundary points of ω, where ω is thin is of harmonic measure zero with respect to ω, and finally fine topology and fine limit are discussed.

M. Ohtsuka (Lawrence, Kans.)

5097:

Hervé, Rose-Marie. Développements sur une théorie axiomatique des fonctions surharmoniques. C. R. Acad. Sci. Paris **248** (1959), 179–181.

Under the same assumptions as in the preceding paper, let C [C^+, resp.] be the set of continuous [and positive, resp.] functions in Ω, with compact support, and H [H^+, resp.] be the set of differences [positive differences, resp.] of two continuous potentials in Ω, which coincide outside some compact set. The first theorem asserts that, given $f \in C^+$ with support S and a neighborhood ω of S, there exists $u \in H^+$ which vanishes outside ω and approximates f uniformly in Ω; the resolutivity of a continuous boundary function follows from this theorem too. Theorem 2: Given a set E and point x, there exists a Radon measure $\mu_x^E > 0$ in Ω such that, for any superharmonic function $v \geqq 0$ in Ω, the extremisation $\hat{R}_v^E(x)$ of v is expressed as $\int v \, d\mu_x^E$. Some relations between μ_x^E and thin sets are derived. Now we replace the axiom 3 by 3′. Then any potential $p(x)$ can be represented in the form $\int w(x) d\nu_p(w)$, where $\nu_p > 0$ is a Radon measure on $\mathscr{E}^+_{x_0, \omega_0}$, supported by extremal potentials, and is called the measure associated with p. Theorem 5: $\hat{R}_p^E(x) = \int \hat{R}_w^E(x) d\nu_p(w)$. Next it is shown that, given an open set $\omega \subset \Omega$, any positive superharmonic function v in ω can be decomposed into $w + w'$, where w and $w' \geqq 0$ are superharmonic in Ω, w is harmonic outside $\bar{\omega}$ and w' is harmonic in ω. It follows that the support of any extremal potential consists of a point. Two more theorems are given: One is on the existence of a positive superharmonic function, which tends to $+\infty$ along the given thin set and which assumes a finite value at the point where the set is thin, and the other is a decomposition theorem of a superharmonic function into a potential and a harmonic function.

M. Ohtsuka (Lawrence, Kans.)

5098:

Beurling, A.; and Deny, J. Dirichlet spaces. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 208–215.

Let X be a locally compact space, $\mathscr{C}$ the space of continuous complex-valued functions on X vanishing at infinity, ξ a positive Radon measure on X such that $\xi(\omega) > 0$ whenever ω is open and not empty. A Hilbert space D, whose elements are (equivalence classes of) locally summable functions on X, is said to be a Dirichlet space if the norm $\|u\|$ satisfies these three axioms: (a) For each compact set K there is a constant A_K such that $\int_K |u| d\xi \leqq A_K \|u\|$ for all u in D; (b) $\mathscr{C} \cap D$ is dense in D and dense in $\mathscr{C}$; and (c) if $u \in D$ and if T is a normalized contraction, then $Tu \in D$ and $\|Tu\| \leqq \|u\|$. Here T is a map of the complex plane into itself which leaves the origin fixed and decreases distances, and Tu is the obvious composition of functions.

A detailed account of Dirichlet spaces over a finite set X is to be found in the authors' paper [Acta Math. **99** (1958), 203–224; MR **20** #5373]. The present note sets forth, largely without proofs, the main properties of general Dirichlet spaces; most of the proofs can be found in an exposition by Deny [#5093 above].

The authors first define pure potentials and state the fundamental theorems concerning "condensers", equilibrium potentials, balayage, as in their paper on finite spaces. They next define the capacity of an open set and the exterior capacity of an arbitrary set. It turns out that in each equivalence class u one can single out a function which is determined up to a set of exterior capacity zero and has certain properties of regularity; this result is the first step to a "fine" theory of potentials.

The problem of explicitly representing the norm of a Dirichlet space is interesting and difficult. In the general situation one can find canonically norms $\|u\|_\lambda$ that increase to $\|u\|$ as $\lambda \searrow 0$, the norms $\|u\|_\lambda$ having the representation

$$\|u\|_\lambda{}^2 = \lambda^{-1} \int [1 - m_\lambda] |u|^2 d\xi + (2\lambda)^{-1} \iint |u(x) - u(y)|^2 d\alpha_\lambda(x, y),$$

where α_λ is a symmetric measure on $X \times X$ and m_λ is the density (relative to ξ) of the projection of α on X. If X is an Abelian group, if ξ is the Haar measure, and if translation by group elements induces a continuous unitary representation of X on D, then the norm can be written explicitly by means of the Fourier transform as

$$\|u\|^2 = \int \lambda(\hat{x}) |\hat{u}(\hat{x})|^2 d\hat{x}, \quad u \in \mathscr{C} \cap D,$$

where λ is a continuous function on $\hat{X}$ such that the form $\sum [\lambda(\hat{x}_i) + \lambda(\hat{x}_j) - \lambda(\hat{x}_i - \hat{x}_j)] \rho_i \bar{\rho}_j$ is positive for all choices of $\hat{x}_1, \cdots, \hat{x}_n$ and such that $1/\lambda$ is locally summable on $\hat{X}$.

The spectrum $\sigma(u)$ of an element u is defined; every element u is the limit of linear combinations of pure potentials having spectrum in $\sigma(u)$.

Dirichlet rings are defined. In such a ring, over a separable space X, each closed ideal is a prime ideal and is the intersection of the maximal ideals containing it.

The note clearly shows that the concept of Dirichlet space captures and generalizes a good part of what goes by the name "potential theory", especially the part involving the notion of energy.

G. A. Hunt (Princeton, N.J.)

5099:

★Brelot, M. Éléments de la théorie classique du potentiel. Les Cours de Sorbonne. 3e cycle. Centre de Documentation Universitaire, Paris, 1959. 191 pp. Paperbound.

La théorie du potentiel s'est développée depuis trente ans dans diverses directions qu'on peut caractériser ainsi: Noyaux-fonction et noyaux-mesure, fonctions harmoniques et surharmoniques, problème de Dirichlet, intégrale de Dirichlet, énergie, probabilités et semi-groupes.

Ces divers aspects sont déjà en germe dans la théorie classique, et on peut tenter de traiter maintenant cette théorie de telle façon que les notations, définitions et démonstrations soient commodément transposables dans les diverses généralisations. C'est ce qu'a fait M. Brelot dans ce travail d'initiation à la théorie moderne du potentiel.

Un appendice, destiné antérieurement à un cours pour étudiants d'Université, développe simplement les notions fondamentales sur les fonctions harmoniques: Principe du maximum, formules de Green, usage du Laplacien, transformation conforme, notions sur le potentiel logarithmique, analyticité des fonctions harmoniques, intégrale de Poisson, convergence des familles de fonctions harmoniques, étude au voisinage d'un point singulier.

Ce rappel de faits élémentaires permet à l'auteur de se placer dès lors à un niveau assez élevé dans les douze chapitres qui constituent son ouvrage. 1. Un premier chapitre rassemble quelques lemmes, dont certains sont nouveaux, qui seront des outils de base pour la suite: Passages à la limite, enveloppe inférieure, norme de Dirichlet, gradient d'une fonction harmonique près d'une frontière régulière. 2. Fonctions surharmoniques: Caractérisation différentielle, opérations, structure locale, approximation par des régularisées, et théorème de convexité de Riesz. Réticulation du cône des harmoniques $\geqq 0$ dans un ouvert. 3. Un court chapitre sur les ensembles polaires. 4. Potentiels classiques dans R^n: On donne le théorème de représentation de Riesz, d'abord local en utilisant les distributions de Schwartz, puis global dans une boule ouverte munie de sa fonction de Green. On montre la régularité du noyau newtonien, puis le théorème d'Evans. 5. Capacités: À tout compact K d'une boule ouverte on associe son potentiel capacitaire de Green W_K par la technique des fonctions surharmoniques. On étudie la continuité à droite et la sous-additivité forte de la fonction $K\to W_K$; d'où des énoncés analogues pour la capacité. On peut alors identifier les ensembles polaires avec ceux de capacité extérieure nulle. Une étude de la notion générale de capacité conduit au théorème de capacitabilité pour ensembles K-analytiques. 6. Potentiels généraux. Le "théorème de convergence", clef de la théorie fine du potentiel, est démontré dans un cadre très général, pour les noyaux $G(x, y)$ semi-continus inférieurement et réguliers sur un espace localement compact. On en donne les applications classiques aux fonctions surharmoniques, et au balayage. 7. L'étude fine des fonctions surharmoniques nécessite la notion d'effilement d'un ensemble en un point; on en donne des critères commodes; et on étudie la topologie fine définie à partir de l'effilement. On montre que l'ensemble des points d'un ensemble où il est effilé est polaire. 8. Le problème de Dirichlet pour un ouvert ω borné de R^n est ici étudié par la méthode de Perron: Une donnée frontière f est dite résolutive si deux fonctions harmoniques H_f et $\bar{H}_f$ associées à f sont égales; les f continues le sont et on peut alors définir la mesure harmonique μ_x associée à un point, et caractériser la résolubilité d'une f par sa μ_x-sommabilité (pour un point x). On définit les points réguliers et irréguliers, et on caractérise ceux-ci comme points d'effilement du complémentaire de ω. 9. On définit la fonction de Green d'un ouvert borné; cet outil nouveau fournit un théorème de représentation globale de Riesz pour toute surharmonique $\geqq 0$. On établit l'important principe de domination, et on étudie le balayage et l'équilibre pour la fonction de Green. 10. On montre simplement le principe de Dirichlet et on l'utilise pour donner une seconde solution au problème de Dirichlet. 11. Ce chapitre expose l'essentiel de la théorie de H. Cartan, basée sur l'énergie. En particulier on montre que E^+ est complet et on interprète le balayage sur un compact comme projection orthogonale dans un espace de Hilbert. On montre enfin l'égalité, à un facteur près, de l'énergie d'une mesure et de l'intégrale de Dirichlet de son potentiel. 12. Le dernier chapitre définit, pour les ouverts bornés, la frontière de Martin, et établit le théorème de représentation (avec unicité) des fonctions harmoniques $\geqq 0$ en termes de fonctions harmoniques extrémales.

Une précieuse bibliographie des travaux de base récents termine chaque chapitre; on y trouve souvent aussi l'indication brève des directions de recherche récentes.

L'auteur a su, en restant dans le cadre restreint du potentiel newtonien de R^n (ou des potentiels de Green pour ouverts bornés), éviter l'écueil d'une technique trop raffinée. Aussi son exposé reste-t-il attrayant pour les chercheurs débutants, tout en leur ouvrant des horizons sur les développements plus récents de la théorie. Pour le spécialiste ce sera un excellent rappel de résultats connus mais jusqu'ici dispersés et démontrés par des méthodes disparates. Cet ouvrage vient donc combler heureusement une lacune et mérite de devenir rapidement un ouvrage de base, à la fois pour le débutant et le spécialiste de la théorie du potentiel. *G. Choquet* (Paris)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

5100:

Murta, Manuel. A rule of symbolic calculus. Rev. Fac. Ci. Univ. Coimbra **27** (1958), 17–21. (Portuguese)

The rule is for transforming from a polynomial in the displacement operator E to one in the differential operator D or to one in the difference operator Δ when the operand is a polynomial. *A. S. Householder* (Oak Ridge, Tenn.)

5101:

Chakrabarti, S. C. A few identities on higher differences. Math. Student **26** (1958), 17–19.

With the notations

$$(a^m)_{k,n} = \prod_{j=0}^{n-1} (a^{m-kj}-1),$$

and ${}^nS_p$ = sum of the products of n factors $1, a, \cdots, a^{n-1}$ taken p at a time (${}^nS_0=1$, ${}^nS_p=0$ if $p<0$ or $>n$), the

author finds the identity

$$\sum_{p=0}^{r}(-1)^p(a^{2r-1-2p})_{2,r-p}(a^{2r-2p}-1)^{2r}S_{2p}=(-1)^{r-1}(a^{2r})_{1,2},$$

and three other identities involving higher differences.
A. Jaeger (Cincinnati, Ohio)

5102:

Evgrafov, M. A. The asymptotic behaviour of a solution of difference equations. Dokl. Akad. Nauk SSSR **121** (1958), 26–29. (Russian)

The author announces eight theorems on the asymptotic behavior of certain linear difference equations in one variable, and of systems of such equations, as the argument grows to ∞. The difference equation is

$$(1)\qquad y(n+k)+\sum_{m=1}^{k}a_m(n)y(n+k-m)=0,$$

where $\lim_{n\to\infty}a_m(n)=a_m$. Define

$$P_n(\lambda)=\lambda^k+a_1(n)\lambda^{k-1}+\cdots+a_k(n)=\prod_{i=1}^{k}[\lambda-\lambda_i(n)].$$

Theorem 1: Assume $\lim_n \lambda_i(n)=\lambda_i\neq 0$ (all i) and that $\lambda_i\neq\lambda_j$ for $i\neq j$. Assume $\sum_{n=1}^{\infty}|a_m(n+1)-a_m(n)|<\infty$ ($m=1,\cdots,k$). Then each solution of (1) has the form

$$(2)\qquad y_m(n)\sim\lambda_m{}^{-1}(1)\cdots\lambda_m{}^{-1}(n)\quad(n\to\infty).$$

If $\lambda_i(n)\to 0$ or $\to\infty$, there is a generalization in Theorem 3: Let $P_{n,m}(\lambda)=P_n(\lambda)[\lambda-\lambda_m(n)]^{-1}$. Suppose $a_k(n)\neq 0$ for $n\geqq 1$, and that

$$\sum_{n=1}^{\infty}\left|\frac{\lambda_i(n)P_{n+1,i}(\lambda_i(n))}{\lambda_j(n)P_{n+1,j}(\lambda_j(n))}\right|<\infty\quad(i\neq j).$$

Then any solution of (1) has the form (2), where

$$y_m(n)\sim\mu_m(1)\cdots\mu_m(n),$$

$$\mu_m(n)=\frac{P_{n+1,m}(\lambda_m(n+1))}{\lambda_m(n)P_{n+1,m}(\lambda_m(n))}.$$

The other results give generalizations to finite systems, to infinite systems, and to integral equations. They are too technical to reproduce.
G. E. Forsythe (Stanford, Calif.)

5103:

Aczél, Johann. Zur Arbeit "Über eine nichtarchimedische Addition und die Frage ihrer Verwendung in der Physik". Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **7** (1957/58), 439–443. (Russian, English and French summaries)

In the paper cited in the above title, M. Strauss [same Z. **5** (1955/56), 93–97; MR **18**, 638] discussed the binary operations (E) $a+b=(a+b)(1+ab)^{-1}$, (G) $a\widehat{+}b=a+b-ab$ over real numbers. He showed that (G) cannot be transformed into (E) by a function g (possessing an inverse), in the sense that $(G)g=(E)$ is impossible. Aczél argues that the transformation of an operation U is usually defined by $g^{-1}Ug$ instead of Ug, and shows that with this definition (G) can be transformed into (E). All such functions g are obtained by solving the functional equation (1) $g(a)+g(b)-g(a)g(b)=g((a+b)(1+ab)^{-1})$. The totality of solutions are given by $g(a)=2a(1+a)^{-1}$; $g(a)=1-(1-a)^k\times(1+a)^{-k}$ $(k\neq 0)$. The author takes issue with Strauss on other points; in particular, he shows that an operation $a\widehat{+}b$ exists with respect to which the open interval $(-1,1)$ is a continuous abelian group with unit element $e=0$, for which

$$(-1)\widehat{+}a=a\widehat{+}(-1)=-1\quad(a\neq 1);$$

$$1\widehat{+}a=a\widehat{+}1=1\quad(a\neq -1);$$

$$a\widehat{+}b=a\widehat{+}b\quad\text{for}\quad a,b\in\{0,1\};$$

$$(-a)\widehat{+}(-b)=-(a\widehat{+}b).$$

This operation $a\widehat{+}b$ is explicitly given, but is lengthy in statement.
I. M. Sheffer (University Park, Pa.)

SEQUENCES, SERIES, SUMMABILITY

See also 4997.

5104:

Perron, Oskar. Über zwei Kettenbrüche von H. S. Wall. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1957**, 1–13 (1958).

The author uses only elementary continued fraction theory to prove the following two theorems. (1) The continued fraction

$$\frac{1}{1+\beta_1-}\,\frac{\alpha_2}{1+\beta_2+\alpha_2-}\,\frac{\alpha_3}{1+\beta_3+\alpha_3-}\cdots$$

converges if $|\alpha_{\nu+1}|\leqq\rho^2$, $|\beta_\nu|\leqq(1-\rho)^2$ $(\nu=1,2,3,\cdots)$, where $0<\rho<1$. (2) The continued fraction

$$\frac{1}{1+\beta_1+\alpha_1-}\,\frac{\alpha_1}{1+\beta_2+\alpha_2-}\,\frac{\alpha_2}{1+\beta_3+\alpha_3-}\cdots$$

converges if $|\alpha_\nu|\leqq\rho^2$, $|\beta_\nu|\leqq(1-\rho)^2$ $(\nu=1,2,3,\cdots)$, provided $0<\rho\leqq\frac{1}{2}$. These continued fractions may be readily transformed into well-known continued fractions whose approximants are, respectively, the even and odd approximants of the continued fraction $K[a_n/1]$. The author thus provides elementary proofs for theorems given by Wall [Proc. Amer. Math. Soc. **7** (1956), 1090–1093; MR **18**, 635] which extended theorems proved by Thron [Bull. Amer. Math. Soc. **49** (1943), 913–916; Duke Math. J. **10** (1943), 677–685; MR **5**, 118], which in turn extended results of the reviewer [ibid. **4** (1938), 775–778]. Thron's review (referred to above) of Wall's paper gives other relevant references.
W. Leighton (Cleveland, Ohio)

5105:

Tsuchikura, Tamotsu. On a local property of absolute Cesàro summability. Sûgaku **7** (1955), 157–159. (Japanese)

In the author's previous paper [Tohoku Math. J. (2) **5** (1954), 302–312; MR **15**, 866], he stated that summability $|C,\frac{1}{2}|$ is not a local property even for a continuous function. The reviewer of this paper said that "there is a discrepancy in the literature which needs clearing up, since Foà [Boll. Un. Mat. Ital. (2) **2** (1940), 325–332; MR **2**, 94] proved that summability $|C,\alpha|$ $(\alpha>1/p)$ is a local property for functions belonging to L^p." The reviewer seems to have missed the correction of Foà [ibid. (2) **3** (1941), 393–394; MR **3**, 105]. The author's

statement is right; however, his argument is complicated and difficult to follow. So he gives here another weaker but easier theorem than the previous one. Employing Rademacher's functions, he shows that summability $|C, \frac{1}{2}|$ is not a local property for a function which belongs to L^p for every $p>1$, and summability $|C, \alpha|$ $(\alpha<\frac{1}{2})$ is not a local property for a continuous function.

G. Sunouchi (Evanston, Ill.)

5106:

Srivastava, Pramila. On the second theorem of consistency for strong Riesz summability. Indian J. Math. **1**, no. 1, 1–16 (1958).

Let $\{\lambda_n\}$ be a sequence of non-negative numbers increasing to infinity. Let $\varphi(t)$ be a non-negative increasing function, defined for $t\geqq 0$ and such that $\varphi(t)\to\infty$ as $t\to\infty$. The second theorem of consistency for Riesz summability asserts that a series which is Riesz summable (R, λ, k) to a given sum is summable $(R, \varphi(\lambda), k)$ to the same sum provided that, roughly speaking, $\varphi(t)$ increases regularly and not much more rapidly than t. The first general theorem of this nature was obtained by Hardy [Proc. London Math. Soc. (2) **15** (1916), 72–88]. His result was generalized by Hirst [ibid. **33** (1932), 234–250] and by Kuttner [J. London Math. Soc. **26** (1951), 104–111; MR **12**, 696]. The author establishes for strong Riesz summability the analogue of Hirst's result. The principal theorem of the paper is as follows: If $\sum a_n$ is summable $[R, \lambda, k]$, then it is summable $[R, \varphi(\lambda), k]$ to the same sum, where $\varphi(t)\uparrow\infty$ and is a $(\kappa+1)$th indefinite integral for $t\geqq 0$ such that (i) if k is an integer, then $\kappa=k$, and

$$(*)\qquad \int^{\omega} t^{\kappa}|\varphi^{(\kappa+1)}(t)|dt = O\{\varphi(\omega)\};$$

(ii) if k is not an integer, then (*) holds with $\kappa=h+1$, where h is the greatest integer less than k, and in addition either $\varphi'(t)$ is a monotonic increasing function or $\varphi'(t)$ is a monotonic decreasing function, $t\varphi''(t)=O\{\varphi'(t)\}$ and $k>1$. A counter-example is given to show that, if $0<k<1$, summability $[R, \lambda, k]$ does not imply summability $[R, \log\lambda, k]$.

J. G. Herriot (Stanford, Calif.)

5107:

Jakimovski, A. Some remarks on Tauberian theorems. Quart. J. Math. Oxford Ser. (2) **9** (1958), 114–131.

Typical of the main results of this paper is the following theorem (stated here for sequences): Let $\{s_n\}$ be A-summable to s, and for some $\lambda>1$, $M>0$, and finite c, $\liminf_{n\to\infty} \min_{n<m\leqq\lambda n} (s_m-Ms_n)\geqq -c$; then

$$Ms-c\leqq \lim_{n\to\infty}\genfrac{}{}{0pt}{}{\inf}{\sup}\, s_n \leqq \frac{s+c}{M},$$

and $\{s_n\}$ is (C, k)-summable to s for each $k>0$. If $(M-1)s=c$, then $\lim_{n\to\infty} s_n=s$. The bounds on $\liminf s_n$ and $\limsup s_n$ are best possible.

The result is a specialization of a theorem on (L, α)-summability. $s(x)$ is (L, α)-summable to s for some fixed $\alpha>-1$ if

$$L^{\alpha}(\tau) = \frac{\tau^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^{\infty-} s(t)t^{\alpha}e^{-\tau t}dt$$

exists for all $\tau>0$ and approaches s as $\tau\downarrow 0$. The (L, α) method is regular, and it is shown that (L, β)-summability implies (L, α)-summability to the same sum for all α such that $\beta>\alpha>-1$. The use of non-absolutely convergent integrals here is a non-specious generalization.

The iteration of the (L, α) method and the continuous Hausdorff transformation is also considered. The continuous Hausdorff transformation of $s(x)$ by means of $\phi(x)$, in short the $(H, \phi(x))$ transform of $s(x)$, is defined by $t(x)=\int_0^1 s(xt)d\phi(t)$, $x\geqq 0$. Here $\phi(x)$ is of bounded variation in $[0, 1]$. It is shown that if $s(x)$ is (L, α)-summable to s for some $\alpha>-1$ and $t(x)$ is any regular $(H, \phi(x))$ transform of $s(x)$, then $t(x)$ is (L, α) summable to s. The tauberian theorems are obtained by examining iterates of the (L, α) transform with particular $(H, \phi(x))$ transforms. The results obtained include as special cases results of Buck [same J. **6** (1955), 128–131; MR **17**, 253] and Vijayaraghavan [J. London Math. Soc. **1** (1926), 275–282].

D. Waterman (Lafayette, Ind.)

5108:

Parameswaran, M. R. Two Tauberian theorems for functions summable (A). J. Indian Math. Soc. (N.S.) **22** (1958), 77–83.

If $s(u)$ $(u\geqq 0)$ has bounded variation in every finite interval and $s(0)=0$, the Laplace and (C, k) transforms of $s(u)$ are defined by

$$L(t) = \int_0^{\infty} e^{-tu}ds(u) \quad (t>0),$$

$$C^k(x) = kx^{-k}\int_0^x (x-u)^{k-1}s(u)du.$$

The main conclusion of the paper is that if $L(t)\to S$ as $t\to+0$ and if (1) $C^k(x)-(1+c)C^{k+1}(x)$ is bounded below as $x\to\infty$ for some real numbers c and $k\geqq 0$, then $C^{k+1}(x)\to S$ as $x\to\infty$. The special case with $c=0$ was proved by Rajagopal [Amer. J. Math. **69** (1947), 371–378; **76** (1954), 252–258; MR **9**, 26; **15**, 522].

The index $k+1$ in the conclusion may be replaced by k if the function (1) is assumed to tend to a limit. The analogue for sequences generalises a theorem of Szasz [Duke Math. J. **1** (1935), 105–111].

H. R. Pitt (Nottingham)

5109:

★Pitt, H. R. Tauberian theorems. Tata Institute of Fundamental Research, Monographs on Mathematics and Physics, 2. Oxford University Press, London, 1958. xi+174 pp. $6.00; 22.50 rs.

This is the first book exclusively devoted to the study of tauberian theorems, which have been associated with the names of Hardy, Littlewood, and Wiener among others. The author has succeeded in giving a brief introduction to some of the more important tauberian theorems, including mercerian theorems as limiting cases, and the methods which have been developed to prove them. Because of the size of the book, he omits, except for occasional relevant references, the theory of ideals in normed rings, which Gelfand and others have used to relate Wiener's general tauberian theorems to general algebraic properties of these structures [e.g., L. H. Loomis, *An introduction to abstract harmonic analysis*, van Nostrand, Toronto-New York-London, 1953; MR **14**, 883].

Chapter I states the definitions and plan of the book, which deals essentially with relations of the form (1) $g(u)=\int k(u, v)s(v)dv$ (integrals without limits hereafter

are over $(-\infty, \infty)$) between the functions $g(u)$, $s(v)$ and the kernel $k(u, v)$. Under certain conditions on $k(u, v)$ it is natural to expect that the boundedness or convergence as $u\to\infty$ of $g(u)$ will follow from the same properties of $s(v)$. Theorems of this kind are called abelian theorems after the theorem of Abel to the effect that the convergence to A of the series $\sum a_n$ implies that (2) $A(r)=\sum a_n r^n \to A$ as $r\to 1-0$. The converse is not true, and it is this fact which forms the basis of the modern theory of summation of divergent series. If we put $g(u)=A(e^{-1/u})$, $s(v)=\sum_{n\le v} a_n$, $k(u, v)=u^{-1}e^{-(v/u)}$, then the transformation from $\sum a_n$ to $A(r)$ assumes the form (1).

Theorems in which conclusions about the behavior of $s(v)$ in (1) are derived from the assumptions about that of $g(u)$ are generally much deeper than the corresponding abelian theorems. And this tract is mainly concerned with the convergence or other order properties of $s(v)$ and $g(u)$ as $u\to\infty$. Theorems of this kind are called tauberian theorems after the theorem of Tauber [Monatsh. Math. **8** (1897), 273–277], which states that the converse of Abel's limit theorem is true in the sense that (2) implies the convergence of $\sum a_n$, provided that the tauberian condition $a_n=o(n^{-1})$ is satisfied.

Chapters II and III give some account of the results which can be obtained by elementary methods. These include all "o" theorems and the classical "O" theorems, the first one of which was proved by G. H. Hardy [Proc. London Math. Soc. (2) **8** (1910), 301–320] and then by J. E. Littlewood [ibid. **9** (1910), 434–448]. There are also theorems in which only the boundedness, and not the convergence of $s(v)$, is deduced from that of $g(u)$ and an "O" tauberian condition. These latter results are of some intrinsic interest and importance in that they are substantially best possible and establish the boundedness of $s(v)$, which appears as a hypothesis in Wiener's theory (chapter IV), and which is the first step in the classical tauberian theorems.

Chapter IV deals with Wiener's theory [Ann. of Math. (2) **33** (1932), 1–100], which transformed the whole subject into a simple form, as may be seen from one of the two forms of his general tauberian theorems:

Suppose that $k_1(x)$, $k_2(x)$ belong to L, and

$$\int \exp(-itx)k_1(x)dx \neq 0$$

for real t, that $s(x)$ is bounded and that

$$\lim_{x\to\infty} \int k_1(x-y)s(y)dy = A\int k_1(x)dx;$$

then

$$\lim_{x\to\infty} \int k_2(x-y)s(y)dy = A\int k_2(x)dx.$$

All the special theorems then known may be expressed in the form above. Wiener's theorems and their extensions are applied to special tauberian theorems such as summation by the methods of Cesáro, Riesz, Lambert, Riemann, Hausdorff, Euler, and Borel, and Abel's summation with a curved path.

Chapter V is occupied with mercerian theorems, after the theorem proved by J. Mercer [Proc. London Math. Soc. (2) **5** (1907), 206–224]. Its integral analogue reads: if $\mathrm{R}(\alpha)>0$ and

$$g(x) = \alpha s(x)+\frac{(1-\alpha)}{x}\int_0^x s(y)dy \to 0 \quad \text{as } x\to\infty,$$

then $s(x)\to 0$. By exponential transformations we have

$$g(e^x) = \alpha s(e^x)+(1-\alpha)\int_0^x s(e^y)e^{-(x-y)}dy = \int_0^x s(e^{x-y})dk(y),$$

where $k(+0)-k(0)=\alpha$, $k'(y)=(1-\alpha)e^{-y}$ for $y>0$. This belongs to the transformation (1), which may be generalized to the form $g(u)=\int s(v)dh_v(u, v)$ with the "Stieltjes kernel" $h(u, v)$. Under certain conditions on $h(u, v)$ the convergence of $g(u)$ is found to imply that of $s(v)$ even without a tauberian condition and, in some cases, with no condition whatever on $s(v)$. The generalization of theorems of this type and their relationship to linear integro-differential equations are mainly due to the author [Proc. Cambridge Philos. Soc. **40** (1944), 199–211; **43** (1947), 155–163; MR **6**, 273; **9**, 40].

Chapter VI. Tauberian theorems are appropriate in the analytical theory of numbers, which is concerned with the behavior of assemblages of whole numbers as the size of these assemblages increases. One of the most celebrated theorems in this field is the prime number theorem of Hadamard and de la Vallée Poussin [Landau, *Primzahlen*, 2nd. ed., ed. by P. T. Bateman, Chelsea, New York, 1953; MR **16**, 904], which asserts that if $\pi(x)$ is the number of primes not exceeding x, then $\pi(x)\log x\sim x$ as $x\to\infty$. In any attempt to prove the theorem, it is at once apparent that the only easily accessible formulae for $\pi(x)$ all involve its means, and that the deduction of the theorem is essentially tauberian in character. The aim of this final chapter is to set out and compare arguments which have been used to prove the prime number theorem. After giving the classical proofs of the prime number theorem, the author discusses the proofs by the Landau-Ikehara theorem [Landau, loc. cit. pp. 917–924], the Lambert tauberian theorem [Wiener, J. Math. Phys. **7** (1928), 161–184], Ingham's method [J. London Math. Soc. **20** (1945), 171–180; MR **8**, 147] and Selberg's method [Ann. of Math. (2) **50** (1949), 305–313; MR **10**, 595; P. Erdös, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 374–384; MR **10**, 595].

S. Ikehara (Tokyo)

APPROXIMATIONS AND EXPANSIONS

See also 5029, 5033, 5100, 5124, 5141.

5110:

Talalyan, A. A. Universal orthogonal series. Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **12** (1959), no. 1, 27–42. (Russian. Armenian summary)

Let $\{S_k\}$ be a sequence of measurable functions defined on the interval $[a, b]$. A measurable function F defined on $[a, b]$ is said to be an upper limit with respect to the sequence $\{S_k\}$ if

(a) $$\lim_{k\to\infty} \text{meas}\,\{E[S_k(x) > \varphi(x)]\cdot E[\varphi(x) < F(x)]\} = 0$$

for any measurable function $\varphi(x)$,

(b) $$\limsup_{k\to\infty} \text{meas}\,\{E[S_k(x) > \chi(x)]\cdot E[F(x) > \chi(x)]\} > 0$$

for any measurable function $\chi(x)$ for which

$$\text{meas}\, E[F(x) > \chi(x)] > 0.$$

The measurable function $G(x)$ is called a lower limit with respect to $\{S_k\}$ if

(α) $\lim_{k\to\infty} \operatorname{meas} \{E[S_k(x) < \varphi(x)]\cdot E[\varphi(x) < G(x)]\} = 0$

for any measurable function $\varphi(x)$,

(β) $\limsup_{k\to\infty} \operatorname{meas} \{E[S_k(x) < \chi(x)]\cdot E[G(x) < \chi(x)]\} > 0$

for any measurable function $\chi(x)$ for which

$$\operatorname{meas} E[G(x) < \chi(x)] > 0.$$

Let $\{\varphi_n\}$ be a complete ortho-normal system in $L^2[a, b]$ (Lebesgue measure) and F and G measurable functions such that $G(x) \leqq F(x)$ a.e. The author shows that there exists a series $\sum_1^\infty a_k\varphi_k(x)$ with $a_k \to 0$ such that if $S_n(x) = \sum_1^n a_k\varphi_k(x)$, then F and G are upper and lower limits respectively with respect to the sequence $\{S_n\}$, and for each measurable function which satisfies the condition $G(x) \leqq f(x) \leqq F(x)$, a.e., there exists a sequence of integers $\{n_i\}$ such that $S_{n_i}(x) \to f(x)$, a.e.

This paper is a continuation of the author's previous work on the subject [same Izv. **10** (1957), no. 3, 17–34; MR **19**, 742]. *A. Devinatz* (St. Louis, Mo.)

5111:

Videnskiĭ, V. S. On the mutual position of the zeros of consecutive polynomials deviating least from zero. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 723–726. (Russian)

In generalisation of the separation property for the zeros of consecutive orthogonal polynomials, the author considers a pair of continuously differentiable functions as follows: (1) in the open interval (a, b) the polynomial $t_n(x)$ has n simple zeros $x_1, \cdots, x_n$, and $t_{n+1}(x)$ has the simple zeros $y_1, \cdots, y_{n+1}$, (2) an arbitrary non-trivial linear combination $\lambda t_n(x) + \mu t_{n+1}(x)$ has no more than $n+1$ zeros in $[a, b]$, (3) $|t_n(x)| \leqq 1$, $|t_{n+1}(x)| \leqq 1$ in $[a, b]$, (4) $t_n(x)$ and $t_{n+1}(x)$ attain respectively $n+1$, $n+2$ times in $[a, b]$ the successive extrema $+1$, -1. Then $a < y_1 < x_1 < y_2 < \cdots < x_n < y_{n+1} < b$, a similar separation property holding for the extrema. The result comes from the author's 1947 thesis. The special case of the zeros of polynomials with least weighted maximum was considered by the reviewer [Proc. Amer. Math. Soc. **7** (1956), 267–270; MR **18**, 126] without, however, imposing differentiability on the weight-function. In the present author's work, the abolition of the differentiability requirement seems to render the inequalities for the zeros no longer strict. *F. V. Atkinson* (Canberra)

5112:

San Juan, Ricardo. Les inégalités de Gorny-Cartan pour des développements asymptotiques. C. R. Acad. Sci. Paris **248** (1959), 3676–3678.

Let $f(z)$ be analytic in a domain D of which 0 and ∞ are boundary points, and continuous in the closure of D, and let $f(z)$ have the asymptotic expansion $\sum a_kz^k$ with bounds m_n, that is, $f(z) = \sum_{k=0}^{n-1} a_kz^k + f_n(z)z^n$ with

$$\max |f_n(z)| = m_n < \infty.$$

The author shows that if D is an angle $|\arg z| < \alpha\pi$, $0 < \alpha < 2$, then

$$m_k < 8(4e)^{(2-\alpha)/k}(n/k)^{(2-\alpha)/k}m_0{}^{1-k/n}m_n{}^{k/n}.$$

R. P. Boas, Jr. (Evanston, Ill.)

FOURIER ANALYSIS

See also 4993, 5105, 5123, 5143.

5113:

Fletcher, Harvey J. Summing of trigonometric series with coefficients which have a periodic factor. Amer. Math. Monthly **65** (1958), 349–351.

$F(x)$ is in S if it can be expressed in open intervals as a finite sum of Ax^le^{bx}, A, b complex, $l \geqq 0$ an integer. If $\sum_1^\infty r(n) \sin nx$ is in S and $f(n)$ is even and periodic, then $\sum_1^\infty r(n)f(n) \sin nx$ is also in S; and similarly for odd $f(n)$ and cosine series. *D. Waterman* (Lafayette, Ind.)

5114:

Tatarčenko, L. P. Beurling's spectrum and some approximation theorems relative to it. Dokl. Akad. Nauk SSSR **124** (1959), 775–778. (Russian)

Let $M_{\langle\alpha\rangle}$ be the Banach space of the measurable functions defined on $(-\infty, +\infty)$ such that

$$F = \sup |F(x)|[\alpha(x)]^{-1} < +\infty,$$

where $\alpha(x)$ is a positive continuous function satisfying some special conditions. $M_{\langle\alpha\rangle}$ is the conjugate of the space $L_{\langle\alpha\rangle}$ of the measurable functions defined on $(-\infty, +\infty)$ such that $\|f\|_\alpha = \int_{-\infty}^{+\infty} \alpha(-x)|f(x)|dx < +\infty$. For $F \in M_{\langle\alpha\rangle}$ let $\hat{T}_\alpha[F]$ be the weak closed space determined in $M_{\langle\alpha\rangle}[=(L_{\langle\alpha\rangle})']$ by the functions $F(x-t)$ $(-\infty < t < +\infty)$. Following A. Beurling [Acta Math. **77** (1945), 127–136; MR **7**, 61] the spectrum B_F of F is the set of the real λ for which $e^{i\lambda x} \in \hat{T}_\alpha[F]$. Let $F \in M_{\langle\alpha\rangle}$ be continuous. Then: (1) If $\alpha(x) = 1 + |x|^m$ $(m \geqq 0)$ there is a sequence

$$S_n(x) = \sum_{k=-n}^{n} P_k(x)e^{i\lambda_kx}$$

(where $\lambda_k \in B_F$, degree of $P_k \leqq m$), such that $S_n(x) \to F(x)$ uniformly on each compact and

$$\sup |S_n(x)|(1+|x|^{m+1/2})^{-1} \leqq C \sup |F(x)|(1+|x|^m)^{-1};$$

(2) if $\alpha(x) = \exp |x|^\beta$ $(0 < \beta < 1)$, for all $\varepsilon > 0$ there is a sequence $S_n(x) = \sum_{k=-n}^{n} P_k(x)e^{i\lambda_kx}$ (where $\lambda_k \in B_F$) such that

$$\sup |F(x) - S_n(x)| \exp(-|x|^{\beta+\varepsilon}) \to 0 \quad (n \to \infty).$$

Different consequences are also given.

C. Foiaş (Bucharest)

5115:

Kahane, Jean-Pierre. Sur la totalité des suites d'exponentielles imaginaires. Ann. Inst. Fourier. Grenoble **8** (1958), 273–275.

L'auteur construit une suite symétrique de densité nulle $\{\lambda_n\}$, telle que $\{e^{i\lambda_nx}\}$ forme un système total sur tout segment. Ceci rectifie un énoncé antérieur de l'auteur [mêmes Ann. **5** (1953–1954), 39–130; MR **17**, 732], et infirme une hypothèse de L. Schwartz relative au "rayon de totalité" d'une suite d'exponentielles imaginaires [Ann. Fac. Sci. Univ. Toulouse (4) **6** (1942), 111–174; MR **7**, 437]. *S. Mandelbrojt* (Paris)

5116:

★Schwartz, Laurent. Étude des sommes d'exponentielles. 2ième éd. Publications de l'Institut de Mathématique de l'Université de Strasbourg, V. Actualités Sci.

Ind., no. 959. Hermann, Paris, 1959. 151 pp. 1800 francs.

The present book is a re-edition of a work published during the last war in two installments [*Étude des sommes d'exponentielles réelles*, Actual. Sci. Ind. no. 959, Hermann, Paris, 1943; Ann. Fac. Sci. Toulouse (4) **6** (1943), 111–176; MR **7**, 294, 437]. It occupies an important place in a certain recent development of harmonic analysis; until this second edition part of it was accessible only with great difficulty. It therefore seems advisable to review the work once again as a whole.

What we have before us is a methodical investigation of the Fourier analysis of functions which are "generated" over some real or complex domain by some given sequence of complex exponentials. The author's techniques are characterized by a simultaneous use of the principles of functional analysis and the theory of analytic functions of exponential type. Out of them, a general method emerges, which we sketch in its broadest possible outlines.

One begins with a convex subset D of the complex plane, and fixes some suitable "natural" space $E(D)$ of functions on D. If D is a segment, $E(D)$ will be the space of all continuous or L_p functions thereon, whilst if D has interior points, $E(D)$ consists of the functions analytic within D. One is interested in the closed proper subspaces $E_\Lambda(D)$ of $E(D)$ which are generated by various sequences of complex exponentials $\{e^{\lambda z}|\lambda \in \Lambda\}$.

First of all, one uses some variant of the Hahn-Banach theorem to get a measure m on D which is orthogonal to the $e^{\lambda z}$, $\lambda \in \Lambda$. The Laplace transform

$$M(s) = \int e^{sz}dm(z)$$

will always be regular and of exponential type in a half plane or the whole plane, and for each $\lambda \in \Lambda$, $M(s)/(s-\lambda)$ will be, to within a multiplicative constant, the Laplace transform of a measure m_λ on D such that $\int e^{\mu z}dm_\lambda(z) = 1$ if $\mu = \lambda$ or 0 if $\mu \in \Lambda$, $\mu \neq \lambda$. Given then any $f \in E_\Lambda(D)$ we can write formally its Fourier series

$$f(z) \sim \sum_\Lambda A(\lambda)e^{\lambda z}, \tag{1}$$

where $A(\lambda) = \int f(z)dm_\lambda(z)$.

In order to study the convergence of (1), it is necessary at first to restrict oneself to certain kinds of sequences Λ of exponents, which I shall call elementary. (The definition of an elementary sequence Λ depends on the domain D under consideration, but always requires the points of Λ to be restricted to some particular part of the complex plane.) Formally, (1) is equal to $\int f(x)dk_z(x)$, where

$$dk_z(x) = \sum_\Lambda e^{\lambda z}dm_\lambda(x). \tag{2}$$

For fixed z, this kernel has a Laplace transform which can be written as a contour integral

$$\frac{M(s)}{2\pi i}\int_\gamma \frac{e^{tz}}{M(t)(s-t)}\,dt, \tag{3}$$

γ being a contour which encloses the points of Λ but not s. If Λ is elementary, this contour integral makes sense for certain values of z. That is, there is a region D_+ of the complex plane (different from D) such that, if $z \in D_+$, one can do the following. 1° Break up the contour γ into an infinite sum of certain partial ones, each enclosing only a finite number of points of Λ. 2° Use lower estimates for functions of exponential type so as to show that the corresponding infinite sum of contour integrals representing (3) converges. (This procedure is like the one used in V1. Bernstein's book on Dirichlet series [*Leçons sur les progrès récents de la théorie des séries de Dirichlet*, Gauthier-Villars, Paris, 1933] which Schwartz acknowledges.) The convergence thus established carries with it that of a certain sequence of partial sums of (2), and thence of (1) for $z \in D_+$. (It is the same sequence of partial sums for any $z \in D_+$ and any $f \in E_\Lambda(D)$.) One says that these series converge with grouping of terms.

For $z \in D_+$, call $f_+(z)$ the sum of the series in (1) (obtained by grouping terms). If $D_+ \supset D$, it is easy to see that f_+ and f coincide on D. But it may happen that $D_+ \not\supset D$, i.e., that series (1) cannot be summed by grouping terms for $z \in D$. In that case, D_+ always at least borders on D, and one can show (sometimes only with difficulty) that the boundary values of f and f_+ on the common boundary of D and D_+ coincide. Provided Λ is elementary, f_+ is always a true extension of f into D_+. $f_+(z)$ is holomorphic in D_+, and if f ranges over a bounded subset of $E_\Lambda(D)$, the $f_+(z)$ form a normal family in D_+. This is a characteristic property distinguishing proper subspaces $E_\Lambda(D)$ from $E(D)$.

The grouping of terms in order to sum (1) is essential, and not due to any defect in the method. Its necessity arises whenever the points of Λ are not well enough separated from each other.

In general, Λ is not elementary, but partitions into a finite number (often two) of elementary sequences Λ_i. In this case, to each Λ_i corresponds a domain D_i such that $\sum_{\Lambda_i} A(\lambda)e^{\lambda z}$ converges (with grouping of terms) to, say, $f_i(z)$ for $z \in D_i$. Each $f_i(z)$ has the same properties in D_i as $f_+(z)$ has in D_+ in the case of elementary Λ. The D_i are distinct, and if D is a segment, all the D_i at least border on D, if they do not contain it. In that case, f is on D equal to the sum of the boundary values of the f_i. (Obvious modifications are made if f is not continuous but only L_p on D.) This is the natural generalization of Abel summability to the Fourier series (1).

Mainly real intervals D are considered in the present book. Instead of studying arbitrary sequences Λ, the author limits himself to two extreme cases which, between them, manifest all the characteristics of the general situation.

Chapter 1 takes up the easier of these two cases, that of real sequences Λ, without finite limit point. Here, the question of the proper inclusion of $E_\Lambda(D)$ in $E(D)$ is governed by Müntz's theorem, which the author first proves for negative Λ. In general, Λ breaks up into two elementary sequences, its positive part Λ_1 and its non-positive part Λ_2. If $D = [A, B]$, D_1 is $-\infty < \operatorname{Re} z < B$ and D_2 is $A < \operatorname{Re} z < \infty$.

Chapter II is devoted to the study of the finite subsets of positive sequences Λ for which the $e^{\lambda z}$ are complete on $(-\infty, 0]$. Letting $\lambda_0 \in \Lambda$ be fixed,

$$M(\lambda_0, N) = \max\{|A(\lambda_0)| : \|\sum_{|\lambda| \leq N} A(\lambda)e^{\lambda z}\| \leq 1\}$$

is considered for various choices of the norm $\|\ \|$ over $(-\infty, 0]$. The asymptotic behavior of $M(\lambda_0, N)$ as $N \to \infty$ is studied; the results are generalizations of S. Bernstein's inequalities for polynomials (case that Λ = positive integers).

In Chapter III the author turns first to purely imaginary sequences Λ and finite real intervals D. In spite of the closeness of this case to the classical theory of ordinary trigonometric series, it is more difficult than the one of Chapter I.

There is no exact criterion involving Λ for the proper inclusion of $E_\Lambda(D)$ in $E(D)$. The author gives various necessary or sufficient conditions for this, bearing on the density of Λ. If $E_\Lambda(D)$ is properly contained in $E(D)$, Λ breaks up into the two elementary sequences Λ_1, its positive imaginary part, and $\Lambda_2=\Lambda\sim\Lambda_1$. D_1 and D_2 no longer contain D, as in Chapter I, but are the upper and lower half planes respectively. The proof of the above-mentioned Abel summability of (1) is quite delicate.

The author finally considers the analogous, but easier case of a purely imaginary Λ with D taken as a thin rectangle lying along the real axis. Here, in contrast, there is a condition on Λ necessary and sufficient for the proper inclusion of $E_\Lambda(D)$ in $E(D)$. The rest of the theory proceeds largely as for the case when D reduces to a segment, and proofs are omitted. Some other cases of convex open D are briefly discussed.

The whole development has obvious applications to the theory of functions, particularly as regards Dirichlet series and ordinary trigonometric series. Some of these are enumerated in Chapters I and III. These two chapters contain introductory paragraphs setting forth, sometimes with proofs, the functional analysis, Fourier transform theory, and entire function theory used in the book. For lower estimates on functions of exponential type there is an appendix, tracing back the required results to theorems in Vl. Bernstein's book [op. cit.]. (Some of Bernstein's proofs are unnecessarily complicated, by the way.)

The footnote on p. 126 will be less disturbing to the reader if he realizes that, under the conditions assumed there, the exponentials considered are certainly complete on any interval of length less than $2A$.

The historical paragraph at the end of Chapter III is very valuable, and shows the relationship of this theory to the theory of functions as a whole, indicating also its more recent developments, not included in the book. The bibliography goes up to 1957.

The book presents a beautiful model of a scientific investigation, and is a joy to read. It should be read not only by those interested in harmonic analysis or Dirichlet series, but also by ordinary graduate students, in order that they may learn good mathematical style and have something to balance the abstractionistic influences to which they are subjected. *P. Koosis* (New York, N.Y.)

5117:

Maravić, Manojlo. Über die $G_\theta{}^\kappa$-Summierbarkeit der verallgemeinerten Fourier-Reihen. Acad. Serbe Sci. Publ. Inst. Math. **12** (1958), 137–146.

Let D be a bounded domain in euclidean n-space with a piecewise smooth boundary $\bar{D}$. Let

$$0<\lambda_1\leqq\lambda_2\leqq\cdots\leqq\lambda_\nu\leqq\cdots\to\infty$$

be the eigenvalues and $\{\phi_\nu(\Pi)\}$ the orthonormalized eigenfunctions of the boundary value problem $\Delta u+\lambda u=0$ in D, $u=0$ on $\bar{D}$. If $f\in L^1(D)$ and $a_\nu=\int_D f\phi_\nu$, then $\sum a_\nu\phi_\nu(\Pi)$ is called the generalized Fourier series of f, i.e., $S(f)$. Given a non-decreasing, unbounded, positive sequence $\{\lambda_\nu\}$, a series $\sum\alpha_\nu$ is said to be $G_\theta{}^\kappa$-summable to s if

$$G_\theta{}^\kappa(\lambda,x)=\sum_{\lambda_\nu\leqq x}(1-e^{(\lambda_\nu-x)x^{-\theta}})^\kappa\alpha_\nu\to s\quad(x\to\infty).$$

Let $f_P(t)=\Gamma(n/2)/2(\pi)^{n/2}\int_{S_t}f$, where the integration is extended over the surface, S_t, of the sphere of radius t about P. Let $V_\mu(x)=x^{-\mu}J_\mu(x)$, where $J_\mu(x)$ is the Bessel function of order μ. If ρ_P is the distance between P and $\bar{D}$, and ρ is such that $0<\rho<\rho_P$, define

$$\alpha_k(x)=\frac{2^{k-(n/2)+1}\Gamma(k+1)}{\Gamma(n/2)}x^{n/2}\times\int_0^\rho t^{n-1}V_{k+n/2}(t\sqrt{x})\{f_P(t)-f(P)\}dt.$$

By $\sigma^k(x)$ is meant the kth Riesz mean of $S(f)$ at P, $\sum_{\lambda_\nu\leqq x}(1-\lambda_\nu/x)^k a_\nu\phi_\nu(P)$.

The author establishes the following results. Theorem 1: If for some integral $k\geqq 0$, $\alpha_0(t)$ is $G_\theta{}^\kappa$-summable to 0 and

$$\sigma^k(x)=\alpha_k(x)+f(P)+o\{x^{-k(1-\theta)}\}\quad(x\to\infty),$$

then for every $\kappa>k$, $S(f)$ is $G_\theta{}^\kappa$-summable to f at P. Theorem 2: Suppose $f\in L^2(D)$, $\kappa>(n+1)/2(2\theta-1)$ with $1/2<\theta<1$. Then $S(f)$ is $G_\theta{}^\kappa$-summable to f at P if and only if for some ρ, $0<\rho<\rho_P$, $\alpha_0(t)$ is $G_\theta{}^\kappa$-summable to 0.

D. Waterman (Lafayette, Ind.)

5118:

Kanno, Kôsi. On the summation of multiple Fourier series. Tôhoku Math. J. (2) **11** (1959), 25–42.

Theorems are obtained relating the behavior of the spherical means of order p of a function of k variables at a point to the behavior at infinity of the spherical Riesz means of order δ of the Fourier series of the function. The theorems involve ranges on the order of convergence of one of the above which imply the convergence of the other.

P. Civin (Eugene, Ore.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 4993, 5109.

5119:

Mainra, V. P. On certain operational images of infinite series. Bull. Calcutta Math. Soc. **50** (1958), 34–52.

The aim of this paper is to obtain a number of formulae of the Poisson type [E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Clarendon Press, Oxford, 1937; p. 60]. Suppose that for a certain kernel $K(x)$

$$g(x)=\int_0^\infty K(xy)f(y)dy.$$

Put

$$F(x)=\tfrac{1}{2}f(0)+\sum_{r=1}^\infty f(rx),\quad G(x)=\tfrac{1}{2}g(0)+\sum_{r=1}^\infty g(rx);$$

then the Laplace transform (p multiplied) of $(1/y)G(1/y)$ is $p\int_0^\infty\psi(pz)F(z)dz$, where

$$\psi(p)=(1/\pi)\int_0^\infty K(pz/2\pi)dz/(1+z^2).$$

This result is obtained in Theorem 1 under conditions which appear to the reviewer to hold only for $f(x)$, $g(x)$, $F(x)$ and $G(x)$ all being zero. Corresponding

formulae are obtained for series of the types

$$\sum (-1)^{r+1} f((2r-1)x),\ \sum f(r^2 x) \text{ and } \sum f((2r-1)^2 x).$$

There are seven theorems, each with a corollary. A number of examples are given illustrating the results. None satisfies the conditions of the theorems. The author states that the theorems can be proved under different assumptions, but does not give these other proofs.

J. L. Griffith (Kensington)

5120:

Mainra, V. P. On certain theorems in operational calculus. Bull. Calcutta Math. Soc. **50** (1958), 123–149.

The transform discussed in this paper is the Watson type transform defined by

$$g(x) = \int_0^\infty \tilde{\omega}_{\mu,\nu}{}^\lambda (xy) f(y) dy,$$

where

$$\tilde{\omega}_{\mu,\nu}{}^\lambda (x) = \int_0^\infty \tilde{\omega}_{\mu,\nu} (xy) J_\lambda(y) y^{1/2} dy,$$

$$\tilde{\omega}_{\mu,\nu}(x) = x^{1/2} \int_0^\infty J_\mu(t) J_\nu(x/t) dt/t$$

[K. P. Bhatnagar, same Bull. **46** (1954), 179–199; MR **17**, 261]. There are eighteen theorems showing relations between $\tilde{\omega}_{\mu,\nu}{}^\lambda$ transforms of Laplace transforms. The last section of the paper is devoted to functions which are self reciprocal under the $\tilde{\omega}_{\mu,\nu}{}^\lambda$ transform.

J. L. Griffith (Kensington)

5121:

Zemanian, Armen H. Some properties of rational transfer functions and their Laplace transformations. Quart. Appl. Math. **17** (1959), 245–253.

Suppose that $Z(s)$ is the rational fraction

$$(s^n + a_{n-1}s^{n-1} + \cdots)/(s^m + b_{m-1}s^{m-1} + \cdots), \quad m > n,$$

and that $Z_q(s) = \int_s^\infty \underset{(q\text{-times})}{\cdots} \int_s^\infty Z(s)\, ds$; then $Z(s)$ is said to be of class k, where $k = m-n$, if $Z_{k-1}(\sigma)$ is positive for positive σ. Also suppose that $Z(s) = \int_0^\infty W(t) e^{-st} dt$. The author finds in terms of $W(t)$ a necessary and sufficient condition that $Z(s)$ should belong to class k. He also finds bounds for $|W(t)|$, when $Z(s)$ is the product of a number of rational fractions of various classes. Finally, the situation when $Z(s+c)$ is in class k is examined.

This paper is related to a previous one by the same author [same Quart. **16** (1958), 273–294; MR **21** #1135].

J. L. Griffith (Kensington)

5122:

Narain, Roop. Some properties of generalized Laplace transform. I. Riv. Mat. Univ. Parma **8** (1957), 283–306.

Let us call

$$\phi(s) = W[f(t), k, m] = s \int_0^\infty (st)^{m-1/2} e^{-st/2} W_{k,m}(st) f(t) dt$$

the generalized Laplace transform of $f(t)$. The author derives formulas for chains of such transformations with an occasional change of variable or other similar modification thrown in. His first result is: If $\phi(s) = W[t^n h(t), k, m]$, $h(s) = W[f(t), l, n]$, and $s^{1-n+m} f(s^m) = W[g(t), \frac{1}{2}, 0]$, then $\phi(s) = \int_0^\infty g(t) W(s, t) dt$, where W is expressed as an integral involving generalized hypergeometric functions. There are altogether seven results with attendant specializations and examples, and they correspond to certain results on Laplace transforms $W[f(t), \frac{1}{2}, 0]$.

A. Erdélyi (Pasadena, Calif.)

5123:

Goldberg, Richard R. Convolution transforms of almost periodic functions. Riv. Mat. Univ. Parma **8** (1957), 307–312.

Let

$$E(s) = e^{bs} \prod_1^\infty [1-(s/a_k)] e^{s/a_k},$$

where $0 < a_1 \leqq a_2 \leqq \cdots$, $\sum_1^\infty a_k^{-2} < \infty$, and let

$$G(t) = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} [E(s)]^{-1} e^{st} ds.$$

The author obtains necessary and sufficient conditions for a function $f(x)$, $-\infty < x < \infty$, to have a representation of the form $f(x) = \int_{-\infty}^\infty G(x-t)\phi(t) dt$ where $\phi(t)$ is almost periodic. Let $f_n(x) = \prod_1^n [1-(D/a_k)] f(x)$. The conditions are: (i) the functions $f_n(x)$ should be uniformly bounded for $-\infty < x < \infty$; (ii) there should exist some almost periodic function $h(x)$ such that whenever τ is an ε translation number of h it is also an ε translation number for each $f_n(x)$.

I. I. Hirschman, Jr. (St. Louis, Mo.)

5124:

Tihonov, A. N.; and Samarskiĭ, A. A. Asymptotic expansion of integrals with slowly decreasing kernel. Dokl. Akad. Nauk SSSR **126** (1959), 26–29. (Russian)

The asymptotic development of the transform defined by

$$I[h, x_0; f] = \frac{1}{h} \int_a^b \omega\left(\frac{x-x_0}{h}\right) f(x) dx$$

is investigated for $h \to 0^+$ and $a < x_0 < b$. The kernel $\omega(\xi)$ is assumed to have the forms $\omega(\xi) = \sum_1^n q_k^\pm \xi^{-k} + \omega_n^\pm(\xi)$ where $\omega_n^\pm(\xi) = O(\xi^{-n-1})$ as $\xi \to \pm\infty$. Since in general $q_1^+ \neq q_1^-$, the integral $\int_{-\infty}^\infty \omega(\xi) d\xi$ does not usually exist. Let $f_k(x) = f(x) - \sum_0^k f^{(k)}(x_0)(x-x_0)^k/k!$, let $\Omega_k(\xi) = \xi^k \omega_k^\pm(\xi)$ according as $\xi > 0$ or $\xi < 0$, and let $\bar{\Omega}_k = \xi^{-1}\Omega_{k+1}$. The following theorem is stated. If (1) $|f| < M$ on (a, b) and $f^{(n+1)}(x_0)$ exists, and (2) $\omega(\xi)$ is absolutely integrable on any finite interval, then

$$I[h, x_0, f] = \sum_0^n (\hat{I}_k \ln h + I_k) h^k + h^n \rho(h)$$

with $\rho(h) \to 0$ as $h \to 0+$, where

$$\hat{I}_k = (q_{k+1}^+ - q_{k+1}^-) f^{(k)}(x_0)/k!,$$

$$I_k = [C_k + q_{k+1}^+ \ln(b-x_0) - q_{k+1}^- \ln(x_0-a)] \frac{f^{(k)}(x_0)}{k!}$$

$$+ q_{k+1}^+ \int_{x_0}^b \frac{f_k(x)dx}{(x-x_0)^{k+1}} + q_{k+1}^- \int_a^{x_0} \frac{f_k(x)dx}{(x-x_0)^{k+1}}$$

$$- \sum_{s=0}^{k-1} \frac{f^{(s)}(x_0)}{s!(k-s)} \left[\frac{q_{k+1}^+}{(b-x_0)^{k-s}} - \frac{q_{k+1}^-}{(x_0-a)^{k-1}}\right];$$

$$C_k = \int_{-1}^1 \Omega_k(\xi) d\xi + \int_1^\infty [\bar{\Omega}_1^+(\xi) + \bar{\Omega}_k^-(-\xi)] d\xi.$$

The terms involving $h^k \ln h$ appear only if $q_{k+1}^+ \neq q_{k-1}^-$ $(k=1, 2, \cdots)$. Special consideration is given the case $q_1=0$. *N. D. Kazarinoff* (Madison, Wis.)

5125:

Gegelia, T. G. Boundedness of singular operators. Soobšč. Akad. Nauk Gruzin. SSR **20** (1958), 517–523. (Russian)

The author studies the operator

$$S\varphi(P) = \int_E \frac{\omega(\theta)\varphi(Q)}{r^n(Q, P)}\, dQ,$$

where E is a bounded domain in Euclidean n-space, $r(Q, P)$ is the distance between Q and P, $\theta=(Q-P)/r(Q, P)$, ω is an essentially bounded measurable function on the unit sphere σ such that $\int_\sigma \omega(Q)d\sigma_Q=0$, and the integral over E is a Cauchy principal value integral. If ρ is a measurable non-negative function on E, then $\varphi(Q)$ is in the class $L_p(E, \rho(Q))$ if

$$\int_E \rho(Q)|\varphi(Q)|^p dQ < \infty.$$

The author proves the following.

Theorem: If $\varphi(Q) \in L_p(E, \rho(Q))$, $\rho(Q)=\prod_1^m r^{\alpha_k}(Q, O_k)$, with $0 \leq \alpha_k < n(p-1)$ for $0 \leq k \leq m_1 \leq m$ and $0 \leq -\alpha_k < n$ for $m_1 < k \leq m$; $\alpha=(\alpha_1, \cdots, \alpha_m)$; $O_k \in \bar{E}$ $(O_k \neq O_j$ for $k \neq j$; $k, j=1, \cdots, m)$; $p>1$; then $S\varphi(Q)$ exists for all Q in E, $S\varphi(Q) \in L_p(E, \rho(Q))$ and

$$\int_E \rho(Q)|S\varphi(Q)|^p dQ \leq C_{\alpha,p} \int_E \rho(Q)|\varphi(Q)|^p dQ.$$

Theorem: If $\varphi(Q) \in L_p(E, \rho(Q))$,

$$\rho(Q) = \prod_1^{m_1} [r(Q, O_k)]^{\alpha_k(p-1)} \prod_{m_1+1}^{m} [r(Q, O_k)]^{-\alpha_k};$$

$0 \leq \alpha_k < n$ $(k=1, \cdots, m)$, $0 \leq m_1 \leq m$, $\alpha=(\alpha_1, \cdots, \alpha_m)$, $O_k \in \bar{E}$ $(O_k \neq O_j$, $k \neq j$; $k, j=1, \cdots, m)$, $p>1$; then the operator

$$S^*\varphi(P) = \frac{1}{\rho_*(P)} \int_E \frac{\omega(\theta)\rho_*(Q)}{r^n(P, Q)}\, \varphi(Q)dQ,$$

where

$$\rho_*(Q) = \prod_1^{m_1} r^{\alpha_k}(Q, O_k) \prod_{m_1+1}^{m} r^{-\alpha_k}(Q, O_k)$$

and the integral, taken in the sense of the Cauchy principal value, exists for all P in E, $S^*\varphi \in L_p(E, \rho)$ and

$$\int_E \rho(Q)|S^*\varphi(Q)|^p dQ \leq C_{\alpha,p} \int_E \rho(Q)|\varphi(Q)|^p dQ.$$

Theorem: If $f(Q) \in L_p(E, \rho^p(Q))$ and $\varphi(Q) \in L_q(E, \rho^{-q}(Q))$, where $\rho(Q)=\prod_1^m r^{\alpha_k}(Q, O_k)$, $0<\alpha_k<n/q$ $(k=1, \cdots, m_1 \leq m)$, $0 \leq -\alpha_k < n/p$ $(k=m_1+1, \cdots, m)$, $p>1$, $1/p+1/q=1$, $O_k \in \bar{E}$ $(O_k \neq O_j$, $k \neq j$; $k, j=1, \cdots, m)$, then

$$\int_E f(P)dP \int_E \frac{\omega(\theta)\varphi(Q)}{r^n(Q, P)}\, dQ = \int_E \varphi(Q)dQ \int_E \frac{\omega(\theta)f(P)}{r^n(Q, P)}\, dP.$$

J. J. Kohn (Waltham, Mass.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 5062.

5126:

Vekua, N. P. A differential boundary problem of linear relationship for several unknown functions in the case of open contours. Soobšč. Akad. Nauk Gruzin. SSR **21** (1958), 513–518. (Russian)

Let L be a set of simple, open, smooth arcs $L_j=a_jb_j$ $(j=1, 2, \cdots, p)$ in the z-plane, having no points in common, the positive direction on L_j being from a_j to b_j. The vector $\phi(z)=(\phi_1, \phi_2, \cdots, \phi_n)$ is said to belong to class $H^{(m)}$ if, near the endpoints a_j, b_j, $|\phi_k(z)| \leq C_k|z-c|^{-\alpha}$ $(0 \leq \alpha < 1)$, where the C_k $(k=1, 2, \cdots, n)$ are constants and c is one of the points a_j, b_j and $\phi^{(m)}(z)$ is piecewise holomorphic in the z-plane cut along L. Denote by $\phi_+^{(k)}(t_0)$ and $\phi_-^{(k)}(t_0)$ $(k=0, 1, \cdots, m)$ the limiting values at t_0 on L of $\phi^{(k)}(z)$ when t_0 is approached from the left and right, respectively. The author determines $\phi(z) \in H^{(m)}$, with finite order at infinity, so that

$$\sum_{k=0}^{m} [A_k(t_0)\phi_+^{(k)}(t_0) + B_k(t_0)\phi_-^{(k)}(t_0)] = g(t_0),$$

where $A_k(t_0)$, $B_k(t_0)$ $(k=0, 1, \cdots, m)$ are given matrices and $g=(g_1, \cdots, g_n)$ is a given vector that satisfy Hölder conditions. The solution is made to depend upon solving a specific singular, integrodifferential equation, a more general case of which was previously treated by the author [Akad. Nauk Gruzin. SSSR. Trudy Tbiliss. Mat. Inst. Razmadze **24** (1957), 135–147; MR **20** #1898].

J. F. Heyda (Cincinnati, Ohio)

5127:

Vekua, N. P. A boundary problem of linear relationship for several unknown functions. Soobšč. Akad. Nauk Gruzin. SSR **22** (1959), 3–8. (Russian)

Let D^+ be a simply-connected region in the z-plane bounded by the simple, closed, smooth contour L. D^+ lies to the left when L is described in a positive sense and the angle which the tangent to L makes with a fixed direction satisfies a (H) Hölder condition. Functions $\alpha_j(t_0)$, $\delta_{k\nu}(t_0)$, $\gamma_{k\nu}(t_0)$ $(j, \nu=1, 2, \cdots, n$; $k=0, 1, \cdots, m)$, defined on L, possess non-vanishing derivatives and satisfy a Hölder condition; furthermore, these functions transform L uniformly into L with $\alpha_j(t_0)$ describing L in the same direction as t_0 while $\delta_{k\nu}(t_0)$, $\gamma_{k\nu}(t_0)$ describe L in the opposite direction. The author finds vectors $\phi(z)$, $\psi(z)$, meromorphic in D^+, such that

$$\phi_j[\alpha_j(t_0)] = \sum_{k=0}^{m} \sum_{\nu=1}^{n} A_{jk\nu}(t_0)\psi_\nu^{(k)}[\delta_{k\nu}(t_0)]$$

$$+ \sum_{k=0}^{m} \sum_{\nu=1}^{n} B_{jk\nu}(t_0)\overline{\psi_\nu^{(k)}[\gamma_{k\nu}(t_0)]} + g_j(t_0) \quad (j = 1, 2, \cdots, n),$$

where the $A_{jk\nu}$, $B_{jk\nu}$, g_j are preassigned in (H) on L and $\psi_\nu^{(k)}(t_0)$ designates the boundary values on L of $d^k\psi_\nu(z)/dz^k$. The work represents a direct extension of a previous paper by the author [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **21** (1955), 169–189; MR **17**, 377] and the methods used are the same.

J. F. Heyda (Cincinnati, Ohio)

FUNCTIONAL ANALYSIS

See also 4982, 4983, 4995, 5020, 5024, 5175.

5128:

Monna, A. F. Ensembles convexes dans les espaces vectoriels sur un corps valué. Nederl. Akad. Wetensch. Proc. Ser. A. **61** = Indag. Math. **20** (1958), 528–539.

The author considers vector spaces E over nontrivially nonarchimedean valued fields K and with real nonnegative-valued seminorms p such that $p(kx)=|k|p(x)$ and $p(x+y)\leq\max(p(x),p(y))$ for all x and y in E and $k\in K$. Motivated by properties of the sets $A_p=[x\in E;p(x)<1]$ and $A_p{}^0=[x\in E;p(x)\leq 1]$ and in analogy with the case presented by real vector spaces, the author defines the property (C) of subsets $S\subset E: x\in S, y\in S, a\in K, b\in K$ such that $|a|\leq 1$, $|b|\leq 1$, imply $ax+by\in S$. A subset $S\subset E$ is called K-convex if $S=x+S^*$ where S^* has property (C); $S\subset E$ is called a K-convex body if $S=x+S^*$ where S^* is K-convex and absorbant (i.e., every $x\in E$ is in kS^* for all $k\in K$ with $|k|\geq r$ for suitable r). Let

$$p_1(y)=\inf_{y\in kA_p}\{|k|\},\quad p_0(y)=\inf_{y\in kA_p{}^0}\{|k|\}.$$

The sets A_p and $A_p{}^0$ are convex bodies and the author investigates the converse question à la Minkowski. The cases where K has a dense set of values leads to more precise results than the case where its valuation is discrete. In the former case a convex body S containing 0 gives rise to a unique seminorm p such that $A_p\subseteq S\subseteq A_p{}^0$, where p is defined like p_1 and p_0 above. The discrete case still gives rise to a seminorm with the preceding inclusion relations, but it is no longer unique. In addition to the results described, the author investigates various questions of K-convexity in abstracto and also in connection with topological spaces E. *G. K. Kalisch* (Minneapolis, Minn.)

5129:

Raimi, Ralph A. Equicontinuity and compactness in locally convex topological linear spaces. Michigan Math. J. **5** (1958), 203–211.

Let E and F be two topological vector spaces and $\mathscr{F}$ a set of continuous linear operators from E into F. The author investigates the condition for $\mathscr{F}$ to be equicontinuous, and obtains: let E' and F' be total linear manifolds of linear functionals on E and F, respectively, and let $\mathscr{F}'$ be the set of adjoint operators of $\mathscr{F}$ from F' into E'. We suppose that $\mathscr{F}'$ is equi-continuous for the Mackey topology and E' has the convex compactness property. Then for the topology of E and F with a neighbourhood system whose polar sets are compact for the Mackey topology of E' resp. F', $\mathscr{F}$ is equi-continuous if and only if $\mathscr{F}'$ is totally bounded by the weak uniformity, considering every $y'\in F'$ as a mapping from $\mathscr{F}'$ into the uniform space E' for the Mackey topology. *H. Nakano* (Sapporo)

5130:

Vulih, B. Z. On the property of intrinsic normality of generalized semi-ordered rings. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 3–15. (Russian)

This paper is an extension of earlier works of the author: (A) [Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 365–388; MR **15**, 328], and (B) [Mat. Sb. (N.S.) **33** **(75)** (1953), 343–358; MR **15**, 328]. For notation and terminology, see the reviews cited. The present paper generalizes the main theorem quoted in the review of (B) to the case in which X is a b-complete vector lattice lacking a unit but having a fundamental system of unit-like elements. The GPO obtained from X as in (B) no longer needs to have a unit, but has a set of "almost units". An example shows that intrinsic normality of X is not necessary. *E. Hewitt* (Seattle, Wash.)

5131:

Domračeva, G. I. Extended K-spaces. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 17–27. (Russian)

This paper deals with the representation of certain vector lattices as spaces of continuous extended-real-valued functions on topological spaces. It is based on a representation theorem of this sort due to B. Z. Vulih [Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 365–388; MR **15**, 328]. Notation and terminology are as in the review cited. A K^--space is a vector lattice in which countable bounded sets have least upper and greatest lower bounds. A compact Hausdorff space Q is called quasi-extremal if the closure of every open F_σ in Q is open. An Archimedean K^--space X with unit is always representable as a subspace of $\mathfrak{C}_\infty(Q)$, where Q is quasi-extremal. The fact that $\mathfrak{C}_\infty(Q)$ itself is a K^--space if and only if Q is quasi-extremal is due to Ogasawara [J. Sci. Hiroshima Univ. Ser. A. **12** (1942), 37–100; **13** (1944), 41–161; MR **10**, 545]. The imbedding of X in $\mathfrak{C}_\infty(Q)$ will depend upon the choice of unit in X. A K^--space X is called extended if there is an imbedding of X in some $\mathfrak{C}_\infty(Q)$ such that the image of X is all of $\mathfrak{C}_\infty(Q)$. It is shown that extendedness does not depend upon the choice of unit in X. Also, some complicated necessary and sufficient conditions are given for X to be extended. *E. Hewitt* (Seattle, Wash.)

5132:

Veksler, A. I. On extension of regular operators in K-lineals. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 39–52. (Russian)

Let X be a vector lattice and Y a complete vector lattice. An additive operator V carrying X into Y such that $V(x)\geq 0$ if $x\geq 0$ is called regular. A component X' of X is any set of the form $\{y:|y|\cap|x|=0$ for all x in a certain subset of $X\}$. Theorem 1: A regular operator U carrying X' into Y is extensible to a regular operator carrying X into Y if and only if there is a regular operator V carrying X into Y such that $V(x)\geq|U|(x)$ for all $x\geq 0$, $x\in X'$. Theorem 2: All regular operators carrying X' into Y are extensible to regular operators carrying X into Y if and only if there is a projection of X onto X'. *E. Hewitt* (Seattle, Wash.)

5133:

Veksler, A. I. Certain properties of normal operators in K-lineals. Leningrad. Gos. Ped. Inst. Uč. Zap. **166** (1958), 53–64. (Russian)

Notation and terminology are as in the preceding review. Special classes of regular operators carrying X into Y are studied. *E. Hewitt* (Seattle, Wash.)

5134:

Schaefer, Helmut. Halbgeordnete lokalkonvexe Vektorräume. Math. Ann. **135** (1958), 115–141.

The paper under review is a systematic study of partially ordered locally convex linear spaces with emphasis on duality. A similar study was independently made by the reviewer [Mem. Amer. Math. Soc. no. 24 (1957); MR **20** #1193] (this shall be referred to as [N]), and there are some overlapping results. There are five sections, and some of the highlights of each section are described below.

In section 1, a cone K in a locally convex linear space E (over the field of real numbers) is called normal if the topology is generated by a family $\{p_\alpha\}$ of pseudo-norms each of which is monotone with respect to the partial ordering induced by K (that is, $p_\alpha(x+y) \leqq p_\alpha(x)$ for each x, y in K and each α), and if the cone K is normal each member of the adjoint E' is the difference of two positive members. If E is a Banach space the normality defined above coincides with the classical normality as defined by Kreĭn [Dokl. Akad. Nauk SSSR **28** (1940), 13–17; MR **2**, 315]. A partially ordered locally convex space has a normal positive cone if and only if it admits a base $\mathfrak{U}$ of the neighborhood system of 0 such that $0 \leqq y \leqq x \in U \in \mathfrak{U}$ implies $y \in U$. Let $\langle F, G \rangle$ be a dual system (i.e., the system consisting of a pair of linear spaces F and G with a bilinear form $\langle\,,\,\rangle$ on $F \times G$). A family $\mathfrak{S}$ of subsets of F is saturated if, whenever $S_1, \cdots, S_n$ are members of $\mathfrak{S}$ and $\rho_1, \cdots, \rho_n$ are positive numbers, each subset of the $\sigma(F, G)$-closed, convex, circled hull of $\rho_1 S_1 \cup \cdots \cup \rho_n S_n$ is again in $\mathfrak{S}$. Let $\mathfrak{S}$ be a saturated family of $\sigma(F, G)$-bounded subsets of F; then a cone K in F is called an $\mathfrak{S}$-cone if the smallest saturated family containing $\{S \cap K : S \in \mathfrak{S}\}$ is equal to $\mathfrak{S}$, or equivalently, for any member S_1 of $\mathfrak{S}$ there is a member S_2 of $\mathfrak{S}$ such that S_1 is contained in the closed, convex, circled hull of $S_2 \cap K$. In the last statement if "closed" is dropped, we have the definition of a strict $\mathfrak{S}$-cone. The notion of an $\mathfrak{S}$-cone is dual to the notion of a normal cone in the sense that, if $\mathfrak{S}$ is a saturated family of subsets in G such that the $\mathfrak{S}$-topology (the topology of uniform convergence on members of $\mathfrak{S}$) is compatible with the dual system $\langle F, G \rangle$ (that is, $\sigma(F, G) \subset \mathfrak{S}$-topology $\subset \tau(F, G)$), a cone K in G is normal with respect to the $\mathfrak{S}$-topology if and only if the dual cone $K' = \{g : \langle x, g \rangle \geqq 0$ for $x \in K\}$ is an $\mathfrak{S}$-cone.

Section 2. In a locally convex space E, a cone K is called a BZ-cone (or strict BZ-cone) if K is an $\mathfrak{S}$-cone (or strict $\mathfrak{S}$-cone) where $\mathfrak{S}$ is the set of all bounded subsets of E. There are various results concerning such cones. The following is typical: Each closed BZ-cone in a semi-reflexive space is a strict BZ-cone. On a sequentially complete bornological space with a strict BZ-cone, each positive linear functional is continuous.

In section 3, the foregoing theory is applied to vector lattices with topologies.

Section 4 deals with various locally convex topologies which are induced by the order structure of a partially ordered linear space E. If E contains an order-unit u (an element u such that, for each $x \in E$, there is a $\lambda > 0$ such that $x \leqq \lambda u$), then the Minkowski functional of $\{x : -u \leqq x \leqq u\}$ can be used to define a norm topology. The author generalizes this procedure to an arbitrary partially ordered linear space and arrives at what he calls the order-topology. In [N], the same topology is defined and is called the "order bound topology". The order-topology is bornological. {Theorem 4.5 asserts that the order topology is always tonnelé, but this is false even for a vector lattice. Some sort of order-completeness condition is needed to make the conclusion true. To theorem 4.2 the hypothesis that the positive cone be closed should be added.} The order topology is stronger than any locally convex topology under which the positive cone is normal. The positive cone is normal under the order topology if the space E has the property that if $0 \leqq y \leqq x_1 + x_2$ and $x_1, x_2 \in K$ then $y = y_1 + y_2$ with $0 \leqq y_i \leqq x_i$ $(i = 1, 2)$; this improves the theorem in [N] in which the same conclusion is obtained under the assumption that E is a vector lattice.

In section 5, it is proved that a partially ordered Hausdorff locally convex space is topologically as well as order isomorphic to a subspace of the space $C(T)$ of all continuous real-valued functions with the topology of uniform convergence on compacta for a suitable locally compact Hausdorff space T if and only if the positive cone is normal (and closed). Finally properties of a basis $\{x_n\}$ of a locally convex space are investigated in terms of the closed conical extension of the set $\{x_n\}$.

I. Namioka (Ithaca, N.Y.)

5135:

Schaefer, Helmut. Halbgeordnete lokalkonvexe Vektorräume. II. Math. Ann. **138** (1959), 259–286.

The present paper, and the review thereof, is a continuation of #5134 above.

In section 6, the author introduces partial ordering and the related notions to complex linear (topological) spaces via their real restrictions. Thus a complex linear space is partially ordered if and only if its real restriction is partially ordered, and a cone K in a complex locally convex space is normal (or an $\mathfrak{S}$-cone) if K is normal (or an $\mathfrak{S}$-cone) in its real restriction. There are two kinds of dual cones: if $\langle F, G \rangle$ is a complex dual system and if K is a cone in F, then the weak dual cone of K is $\{g : g \in G$ and $\operatorname{Re}\langle f, g \rangle \geqq 0$ for all f in $K\}$ and the strong dual cone is $\{g : g \in G$, $\operatorname{Re}\langle f, g \rangle \geqq 0$ and $\operatorname{Im}\langle f, g \rangle \geqq 0$ for all f in $K\}$. These notions enable the author to state some of the earlier results and the forthcoming ones for real or complex spaces.

The convergence of directed subsets of partially ordered locally convex spaces is considered in section 7. It is proved that, if a directed subset H of a partially ordered locally convex space converges weakly to x (in the sense of Moore-Smith, using H itself as the index set), then it converges to x relative to the given topology provided the positive cone is normal. The classical Dini's theorem is a consequence of this theorem, which in turn can be proved using Dini's theorem. The author gives a simple direct proof; however his insistence on filters makes the theorem and the proof less transparent. There are more results of this nature.

The second part of the paper is devoted to the study of positive linear maps. A linear map T on a partially ordered space E into another such space F is called positive if T carries the positive cone K in E into the positive cone H in F. If, in addition, E and F are locally convex spaces, the space of all continuous linear maps is denoted by $\mathfrak{L}(E, F)$ (or simply $\mathfrak{L}$) and the set of all positive members of $\mathfrak{L}$ by $\mathfrak{K}$. If $\mathfrak{S}$ is a saturated class of bounded subsets of E, the space $\mathfrak{L}(E, F)$ with the topology of uniform convergence on members of $\mathfrak{S}$ is denoted by $\mathfrak{L}_{\mathfrak{S}}(E, F)$.

Section 8 contains many results concerning the space $\mathfrak{L}$ and the cone $\mathfrak{K}$ which are analogous to or generalizations of the results on the dual E' and the dual cone K'. For

instance, if K is an $\mathfrak{S}$-cone in E and H is normal in F, then $\mathfrak{K}$ is normal in $\mathfrak{L}_{\mathfrak{S}}$. Or, excluding certain trivial cases, $\mathfrak{L}=\mathfrak{K}-\mathfrak{K}$ implies that $F=\bar{H}-\bar{H}$ and $E'=K'-K'$, and conversely the last two equations imply that $\mathfrak{K}-\mathfrak{K}$ is dense in $\mathfrak{L}$ relative to the topology of simple convergence.

Section 9 deals with the continuity of positive transformations and the convergence of a directed family of linear maps. Here the results are mostly generalizations of earlier ones.

Section 10 gives the main theorem concerning the eigenvalues of positive compact operators, of which the theorem of Krein and Rutman [Uspehi Mat. Nauk (N.S.) **3** (1948), 3–95; Amer. Math. Soc. Transl. no. 26 (1950); MR **10**, 256] is an easy consequence.

Let E be a locally convex space with a topology $\mathfrak{T}$ and let K be a cone in E. A positive linear operator T of $E_K=K-K$ into itself is called K-compact if T is $\mathfrak{T}$-continuous on K and, for some neighborhood U of 0, $T[U\cap K]$ is relatively compact. (One can assume that $U=\{x: p(x)\leq 1\}$ for some pseudo-norm p.) For such a T, the K-spectral radius r_K can be defined by $\lim_{n\to\infty}\sqrt[n]{p(T^n)}$ where $p(T^n)=\sup\{p(T^nx): x\in U\cap K\}$. The theorem states that if K is closed and proper (i.e. $K\cap(-K)=\{0\}$) and if T is a K-compact operator with positive r_K, then r_K is an eigenvalue of T with an eigenvector in K.

The last section contains some theorems on positive operators on Banach and Hilbert spaces. To cite one: let E be a Banach space and let K be a normal BZ-cone in E; then for each continuous positive operator T of E into itself the spectral radius is in the spectrum of T. The proof depends on another theorem concerning the spectral radius of a positive element in a partially ordered Banach algebra. *I. Namioka* (Ithaca, N.Y.)

5136:

Weston, J. D. Relations between order and topology in vector spaces. Quart. J. Math. Oxford Ser. (2) **10** (1959), 1–8.

A partially ordered linear topological space is called an A^*-space if it admits a local base of neighborhoods of 0 consisting of sets of the form $\{x: -u\leq x\leq u\}$. An A^*-space is necessarily locally convex. In the present paper, various characterizations and properties of A^*-spaces are given, of which the following are samples. A partially ordered linear space E can be given a topology under which it is an A^*-space if and only if E has an order-unit u [see #5134 above], and, if this is the case, such topology is unique and is given by the Minkowski functional of the set $\{x: -u\leq x\leq u\}$. There is a positive linear functional on an A^*-space if and only if the origin is not an order-unit. Let E be an A^*-space in which the origin is not an order-unit, and let u be an order-unit. Put $P_u=\{\varphi: \varphi\geqq 0$ and $\varphi(u)=1\}$; then the null-space $\{x: \varphi_0(x)=0\}$ of an extreme point φ_0 of P_u is called a critical hyperplane, and this notion is independent of the choice of u. It is proved that any non-zero frontier point of the positive cone is on some critical hyperplane. *I. Namioka* (Ithaca, N.Y.)

5137:

Rutman, M. A. Operator equations in partially ordered spaces and some qualitative theorems for linear partial differential equations. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1 (73), 234–238. (Russian)

A lecture delivered at the All-Union conference on functional analysis and its applications (January, 1956). The results discussed are essentially contained in articles published earlier by the author [Dokl. Akad. Nauk SSSR **101** (1955), 217–220, 993–996; MR **16**, 1126, 1113].

D. F. Harazov (RŽMat **1957** #7182)

5138:

Shimogaki, Tetsuya. A generalization of Vainberg's theorem. II. Proc. Japan Acad. **34** (1958), 676–680.

In the first part of this paper the author discussed the notion of a splitable operator H of a modulared semi-ordered linear space into itself [Proc. Japan Acad. **34** (1958), 518–523; MR **20** #7231]. In this paper the author generalizes his results to operators of a modulared semi-ordered linear space R into a modulared semi-ordered linear space R' which are almost similar to one another (this means that there exists a set C in R and a set C' in R' and that C and C' both have the following properties: (i) for any $a\in R$ $[R']$, there exists $\alpha>0$ such that $\alpha a\in C$ $[C']$; (ii) $a\in C$ $[C']$ and $|b|\leq|a|$ implies $b\in C$ $[C']$; $a, b\in C$ $[C']$, $|a|\cap|b|=0$ implies $a+b\in C$ $[C']$; and there exists a one to one mapping ϕ of C onto C' such that $\phi(-a)=-\phi(a)$ for all $a\in C$ and $a\leq b$ if and only if $\phi(a)\leq\phi(b)$). *W. A. J. Luxemburg* (Pasadena, Calif.)

5139:

Singer, Ivan. Sur un dual du théorème de Hahn-Banach et sur un théorème de Banach. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 443–446.

L'autore estende alcuni risultati ben noti di Banach al caso di uno spazio di Banach non riflessivo. Il risultato tipico è il seguente. Sia E uno spazio di Banach, e sia M un sottospazio dello spazio coniugato E^*, chiuso nella topologia debole indotta su E^* da E (cioè "régulièrement fermé" nel senso di Bourbaki, oppure "weak*-closed" secondo il termine inglese). Sia $\|x_0\|_M=\inf_{f\in M}|f(x_0)|/\|f\|$ per x_0 in E. Fissato x_0, si consideri il sottospazio

$$G=\{x\in E \,|\, f(x)=f(x_0) \text{ per ogni } f \text{ in } M\}.$$

Teorema: Si ha $\inf_{x\in G}\|x\|=\|x_0\|_M$.

G.-C. Rota (Cambridge, Mass.)

5140:

Day, Mahlon M. On criteria of Kasahara and Blumenthal for inner-product spaces. Proc. Amer. Math. Soc. **10** (1959), 92–100.

The purpose of this paper, as stated by the author, is "to give related generalizations of two quite distinct recent generalizations of the Jordan-von Neumann characterizations of inner-product spaces [Jordan and von Neumann, Ann. of Math. (2) **36** (1935), 719–723]. In one of the "generalizations" referred to [Kasahara, Proc. Japan. Acad. **30** (1954), 846–848; MR **16**, 1032] the class of normed linear spaces is the comparison class, while in the other [Blumenthal, Pacific J. Math. **5** (1955), 161–167; MR **16**, 1139] the characterization is with respect to the class of complete, metrically convex, externally convex metric spaces. {Reviewer's note: In the reviewer's opinion, only the second of the two papers cited gives a generalization of the Jordan-von Neumann criterion, since it characterizes inner-product spaces among a considerably wider class of spaces than normed linear spaces by means of a criterion which reduces to that of Jordan-von

Neumann in the case of normed linear spaces. Kasahara's criterion is a weakening rather than a generalization of the earlier one of Jordan-von Neumann, and the paper under review offers a priori weakenings of the criteria of Kasahara and Blumenthal.} Kasahara showed that the Jordan-von Neumann condition

$$\|f+g\|^2+\|f-g\|^2 = 2(\|f\|^2+\|g\|^2)$$

can be replaced by the weaker assumption that there exists a positive number $\alpha \leqq \frac{1}{2}$ such that for each pair of points x, y a number $\lambda=\lambda(x, y)$ exists, with $\alpha \leqq \lambda \leqq 1-\alpha$, and

$$\lambda\|x\|^2+(1-\lambda)\|y\|^2 \geqq \lambda(1-\lambda)\|x-y\|^2+\|\lambda x+(1-\lambda)y\|^2.$$

The author shows that Kasahara's assumption can be replaced by the following a priori weaker condition: For each pair of points x, y with $\|x\|=\|y\|=1$, numbers λ, μ exist (depending on x, y), $0<\lambda<1$, $0<\mu<1$, such that

$$[\lambda+\mu-2\lambda\mu][\lambda\mu\|x\|^2+(1-\lambda)(1-\mu)\|y\|^2] \sim \mu(1-\mu)\|\lambda x+(1-\lambda)y\|^2+\lambda(1-\lambda)\|\mu x-(1-\mu)y\|^2$$

where $\sim$ stands for one of the relations $=$, $\geqq$, $\leqq$. The purely metric generalization of the Jordan-von Neumann condition due to the reviewer, referred to above, replaces it by its metric essence in the environment of complete, metrically convex, externally convex, metric spaces M, and assumes that each four points p, q, r, s of the space are congruently imbeddable in the euclidean plane provided $pq=qr=\frac{1}{2}pr$. It was then shown that every space M satisfying this feeble euclidean four-point assumption has every finite-dimensional subspace congruent with a euclidean space, and hence is an inner product space. The author shows that the feeble four-point property is implied by the a priori weaker property (the queasy four-point property) according to which to each two distinct points p, r of the space there is a point q metrically between p and r such that for every point s the quadruple p, q, r, s is congruently imbeddable in the euclidean plane. It then follows from the reviewer's result that the space is an inner-product space. *L. M. Blumenthal* (Columbia, Mo.)

5141:

Hirschfeld, R. A. On best approximations in normed vector spaces. II. Nieuw Arch. Wisk. (3) **6** (1958), 99–107.

The author continues his study of best approximation [C. R. Acad. Sci. Paris **246** (1958), 1485–1488; Nieuw Arch. Wisk. (3) **6** (1958), 41–51; MR **20** #2602, #4768] by seeking to characterise complex Hilbert space in terms of its best approximation operators. We retain the notation of the reviews just cited.

Two characterisations are given. Theorem 1: Let E be a complex normed vector space of dimension $\geqq 3$, and suppose that $\|A_G x\| \leqq \|x\|$ for a suitable choice of $A_G x$ and for every two-dimensional subspace G of E. Then E is a (possibly non-separable and incomplete) Hilbert space. (The operator A_G is not generally single-valued.) Theorem 2: Let E be a complex strictly normed vector space of dimension $\geqq 3$, and suppose that A_G (which is single-valued because of the strict norm) is additive. Then E is a Hilbert space. The proofs use several results concerning the characterisation of Hilbert space which appear as exercises set by Bourbaki [*Espaces vectoriels topologiques*, Chap. III–V, Actualités Sci. Ind. no. 1229, Hermann, Paris, 1955; MR **17**, 1109; pp. 138 et seq.].

The following problem is proposed. Let E be a reflexive strictly convex Banach space, so that A_G is defined and single-valued for each G. Take two closed vector subspaces G_1 and G_2 and let G be the closed vector subspace they generate. Put $B_1=I-A_{G_1}$, $B_2=I-A_{G_2}$ and $B=I-A_G$. E is said to admit the "alternating method" if, for all G_1 and G_2, the sequence $B_1, B_2B_1, B_1B_2B_1, \cdots$ converges strongly to B. It is known that Hilbert space admits the alternating method. Does this property characterise Hilbert space? *R. E. Edwards* (Reading)

5142a:

★**Гельфанд, И. М.; и Шилов, Г. Е. Пространства основных и обобщенных функций. [Gel'fand, I. M.; and Šilov, G. E. Spaces of fundamental and generalized functions.]** Obobščennye funkcii, Vypusk 2. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 307 pp. 10.05 rubles.

5142b:

★**Гельфанд, И. М.; и Шилов, Г. Е. Некоторые вопросы теории дифференциальных уравнений. [Gel'fand, I. M.; and Šilov, G. E. Some questions in the theory of differential equations.]** Obobščennye funkcii, Vypusk 3. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 274 pp. 9.05 rubles.

[Tome I: MR **20** #4182.]

Tome II. Chap. I: Espaces vectoriels topologiques. Définitions générales. Espaces de Fréchet et leurs duals. Les A. appellent "espace parfait" un espace de Fréchet où tout ensemble fermé borné est compact. Propriétés standard de ces espaces. Chap. II: fonctions fondamentales (test functions) et fonctions généralisées. Les A. introduisent les espaces $\mathscr{D}$ (ici noté K) et S de L. Schwartz [*Théorie des distributions*, t.I, II, Paris, Hermann, 1950 et 1951; MR **12**, 31, 833]. Quelques autres exemples simples. Les A. introduisent ensuite (§2) les espaces $K(M_p)$ et $Z(M_p)$: soit M_p, $p=0, 1, \cdots$, une suite de fonctions définies dans R^n, avec $1 \leqq M_0(x) \leqq M_1(x) \leqq \cdots$, les $M_p(x)$ étant finis ou tous infinis. Alors $K(M_p)$ est l'espace des fonctions φ indéfiniment différentiables, telles que $M_p(x)D^q\varphi(x)$, $|q| \leqq p$, soit continue et bornée sur R^n; on pose $\|\varphi\|_p=\sup_{|q|\leqq p} M_p(x)|D^q\varphi(x)|$. Cet espace est complet. Si l'on suppose que la condition suivante a lieu: (P) pour tout $\varepsilon>0$, et pour tout p, il existe $p'>p$ et N tels que $|x|>N$, $M_p(x)>N$ entraîne $M_p(x) \leqq \varepsilon M_{p'}(x)$; alors $K(M_p)$ est parfait. L'espace $Z(M_p)$ est construit de façon quelque peu analogue mais à partir de fonctions entières (cf. p. 114 et pp. 124, 125). Le § 3 (p. 126) étudie les duals des espaces précédents. Théorèmes de structure des éléments de ces espaces.

Le chap. III étudie la transformation de Fourier. Les A. reprennent l'exposé du t.I du présent ouvrage, en le complétant: étude du produit de composition, de la transformation de Fourier-Borel, démonstration du théorème de Paley-Wiener et Paley-Wiener-Schwartz. Signalons une remarque intéressante (Vilenkin) sur la transformation de Hilbert (p. 184 et sq.). Ce chapitre est une sorte de résumé, parfois très bref, de la deuxième partie du tome 2 de l'ouvrage de L. Schwartz, loc. cit.

Le chap. IV étudie les espaces du type S et la façon dont la transformation de Fourier y opère. Il s'agit d'un exposé des résultats des A. [Uspehi Mat. Nauk (N.S.) 8 (1953),

no. 6 (58), 3–54; Amer. Math. Soc. Transl. (2) **5** (1957), 221–274; MR **15**, 867; **18**, 736], complétés par les résultats obtenus depuis par les A. eux-mêmes et Levin, Babenko, Gurevitch et d'autres. Voir aussi un exposé en français dans Gel'fand et Šilov, J. Math. Pures Appl. **35** (1956), 383–413 [MR **18**, 493].

Tome III. Le chap. I reprend en détail un appendice du dernier chap. du tome II. Il s'agit d'un exposé complet d'un travail de Gurevitch [Thèse, 1956], généralisant les résultats du chap. IV, t.II. Les chapitres suivants donnent des applications, nombreuses et intéressantes, des résultats obtenus (et surtout ceux du chap. IV, t.II).

Chap. II. Classes d'unicité des solutions du problème de Cauchy. On cherche le vecteur $u=(u_1, \cdots, u_m)$, avec $(\partial/\partial t)u=P(i(\partial/\partial x))u$, $u(x,0)=u_0(x)$, où $P(i(\partial/\partial x))$ est une matrice (m, m), à coefficients opérateurs différentiels en x, à coefficients constants. Formellement, $u(x,t)=\exp(tP(i(\partial/\partial x))u_0(x)$ (méthode opérationnelle), où, $v(s,t)$ désignant la transformée de Fourier en x (en un sens convenable) de $u(x,t)$, $v(s,t)=\exp(tP(s))v_0(s)$. Les A. précisent les deux méthodes. Considérons $Q(s,t_0,t)=\exp(t-t_0)P(s)$; soit p le degré maximum des polynomes intervenant dans P; on a (§ 6; Šilov, Gel'fond, Vorok) $\exp(t-t_0)\Lambda(s)\leq\|Q(s,t_0,t)\|\leq c(1+|s|)^{(m-1)p}\exp(t-t_0)\Lambda(s)$, $t>t_0$, où $\Lambda(s)=\max_k \operatorname{Re}\lambda_k(s)$, les $\lambda_k(s)$ étant les valeurs propres de $P(s)$; on a $\Lambda(s)\leq b|s|^{p_0}$, $p_0\leq p$. Le meilleur p_0 est appelé l'ordre réduit du système. Théorème: si $p_0>1$ [resp. $p_0=1$] il y a unicité du problème de Cauchy dans la classe des vecteurs dont les composantes vérifient $|f(x)|\leq C\exp(b|x|^{p_0'})$, $1/p_0+1/p_0'=1$ [resp. p_0' fini quelconque]. Si $p_0<1$, il y a unicité sans restriction de croissance à l'infini. Les A. démontrent ce théorème au § 3, par la méthode opérationnelle, et au § 4, si $p_0\geq 1$, par la transformation de Fourier (qui, si $p_0<1$, donne un résultat moins bon). Les A. donnent au § 7 des théorèmes du type Phragmén-Lindelöf (Šilov).

Chap. III. Classes où le problème de Cauchy est bien posé.

Le système est dit parabolique si pour σ réel, $\Lambda(\sigma)\leq -c_1|\sigma|^h+c_2$, $c_1>0$, $h>0$; h s'appelle l'exposant de parabolicité. Un système parabolique au sens de Petrowsky l'est au sens précédent, avec $h=p$. D'une variante du théorème de Phragmén-Lindelöf (cf. t.II, chap. IV, p. 252), dont la démonstration est due à Levin, résulte que dans le domaine H_μ:

$$H_\mu = \{s|s=\sigma+i\tau, |\tau| \leq K(1+|\sigma|)^\mu, \mu \geq 1-p_0+h\},$$

on a: $\|Q(s,0,t)\|\leq C\exp(-a't|\sigma|^h)$.

La borne supérieure des μ tel que ceci ait lieu est appelé le genre du système. Théorème: soit $\mu>0$; on suppose que les composantes de u_0 sont dans la classe $K(p_1, b_0)$ des fonctions f avec $|f(x)|\leq C\exp(b_0|x|^{p_1})$, $p_1=p_0/(p_0-\mu)$. Alors, si $b_1>b_0$, le problème de Cauchy est bien posé dans la classe $K(p_1, b_1)$ et dans un intervalle $[0, T]$, T assez petit. Résultat du même type si $\mu\leq 0$ (p. 143).

Le système est hyperbolique [resp. correct au sens de Petrowsky] si $\Lambda(s)\leq a|s|+b$, $\Lambda(\sigma)\leq C$ [resp. $\Lambda(\sigma)\leq C$]. Résultats habituels dans le premier cas. La condition $\Lambda(\sigma)\leq C$ est nécessaire pour que le problème de Cauchy soit bien posé dans l'espace L^1 en x. On introduit le genre; résultats du type de ceux énoncés ci-dessus pour les paraboliques. Même chose pour les systèmes dits "corrects", i.e. tels que $\Lambda(\sigma)\leq c|\sigma|^h+c_1$, $h<1$.

Chap. IV. Décomposition spectrale en fonctions propres généralisées. Soit Φ un espace parfait, H un espace de Hilbert, avec $\Phi\subset H\subset\Phi'$, Φ' dual de Φ, chaque espace étant dense dans le suivant. On désigne par (φ,ψ) le produit scalaire sur H. On donne A opérateur linéaire de Φ dans lui-même, symétrique, i.e. $(A\varphi,\psi)=(\varphi,A\psi)$, $\varphi,\psi\in\Phi$. Alors $A\in\mathscr{L}(\Phi;\Phi)$ et se prolonge par continuité en un opérateur, encore noté A, élément de $\mathscr{L}(\Phi';\Phi')$. Supposons que A, opérateur non borné dans H, de domaine Φ, admette un prolongement hypermaximal (encore noté A) dans H. Alors $A=\int\lambda dE_\lambda$, E_λ famille spectrale de A. Pour e fixé dans H, la fonction $\lambda\to E_\lambda e$ est faiblement (ou scalairement) à variation bornée à valeurs dans Φ'. On dit que l'espace Φ est nucléaire (au sens de Gel'fand-Šilov; les A. n'étudient pas ici les relations avec la nucléarité au sens de Grothendieck) si toute fonction f faiblement à variation bornée à valeurs dans Φ' y est fortement à variation bornée (i.e.: $\Phi=\bigcap\Phi_p$; alors $\Phi'=\bigcup\Phi_p'$; Φ_p est un espace de Banach; f est fortement à variation bornée si elle est à valeurs dans un Φ_p', p fixe convenable, et est dans cet espace fortement à variation bornée). Supposons donc Φ nucléaire au sens précédent. D'après un théorème de Dunford-Gel'fand-Pettis (§ 2) il en résulte que, si $\sigma(\lambda)=(E_\lambda e, e)$, e fixé dans H, la fonction $\lambda\to E_\lambda e$ est σ-presque partout dérivable par rapport à σ, $dE_\lambda e/d\sigma_\lambda=\chi_\lambda$ (§ 3, Th. 1), tout ceci à valeurs dans Φ', et $A\chi_\lambda=\lambda\chi_\lambda$ (§ 4, Th. 1); χ_λ est une fonction propre généralisée de A. Soit $H(e)$ l'espace fermé engendré dans H par les $E_\lambda e$. Alors, tout $\varphi\in H(e)\cap\Phi$ s'écrit: $\varphi=\int\overline{(\chi_\lambda,\varphi)}\chi_\lambda d\sigma(\lambda)$, et $\|\varphi\|^2=\int|(\chi_\lambda,\varphi)|^2d\sigma(\lambda)$ (§ 3, Th. 2). Applications diverses. Structure des χ_λ, lorsque $H=L^2$ (§ 6), et (§ 7) lorsque A est un opérateur différentiel elliptique.

J. L. Lions (Nancy)

5143:

Bochner, S. Generalized conjugate and analytic functions without expansions. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 855–857.

L'auteur généralise très simplement, par voie axiomatique, le théorème classique de M. Riesz sur les normes dans L^p de deux fonctions conjuguées. Ici les L^p sont relatifs à une mesure de Lebesgue $d\sigma$; soit $F=\{f_1+if_2\}$ un anneau de fonctions fermé pour l'involution $f_1+if_2\to f_1-if_2$, dense dans L^p $(1<p<\infty)$; Γ un sous-anneau de F fermé pour l'involution; $\Phi=\{\varphi_1+i\varphi_2\}$ et $\overline{\Phi}=\{\varphi_1-i\varphi_2\}$ des sous-anneaux, tels que $\int\varphi d\sigma=0$ pour tout $\varphi\in\Phi$, et $F=\Gamma+\Phi+\overline{\Phi}$; alors $F=\Gamma\oplus\Phi\oplus\overline{\Phi}$; on suppose que l'application canonique A de F sur Γ satisfait $\|Af\|_p\leq M_p\|f\|_p<\infty$ pour $1<p<\infty$; alors l'application canonique B de F sur Φ satisfait $\|Bf\|_p\leq N_p\|f\|_p<\infty$ pour $1<p<\infty$. D'autres généralisations axiomatiques sont indiquées, de théorèmes de Helson et Lowdenslager [Acta Math. **99** (1958), 165–202; MR **20** #4155].

J. P. Kahane (Montpellier)

5144:

★Красносельский, М. А.; и Рутицкий, Я. Б. Выпуклые функции и пространства Орлича. [**Krasnosel'skiĭ, M. A.; and Rutickiĭ, Ya. B. Convex functions and Orlicz spaces.**] Problems of Contemporary Mathematics. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 271 pp. 9.20 rubles.

In this book, the authors attempt to develop a complete theory of Orlicz spaces and to investigate the continuity of some operators from one Orlicz space into another. The first chapter is devoted to a classification of convex

functions convenient for classifying Orlicz spaces; here we find many new facts about convex functions. In the second chapter, the authors explain many properties of Orlicz spaces, which are considered on Euclidean spaces and are hence separable. Here the authors quote the book: H. Nakano, *Modulared semi-ordered linear spaces*, [Maruzen, Tokyo, 1950; MR **12**, 420], and says that the modulared semi-ordered linear spaces are related to Orlicz spaces. However, they seem not to be acquainted with the fact that Orlicz spaces are completely contained in modulared semi-ordered linear spaces as special examples. Most of the facts in this chapter about Orlicz spaces have already been obtained as properties of modulared semi-ordered linear spaces. Especially, the norm called by the name Luxemburg, related to W. A. J. Luxemburg's thesis [Technische Hogeschool te Delft, 1955; MR **17**, 285], is defined as the second norm in Nakano's book.

In the third and fourth chapters the authors investigate the condition for certain operators from one Orlicz space into another to be continuous or totally continuous. They discuss it successively for linear integral operators A: $Au(x)=\int K(x,y)u(y)dy$; for simple non-linear operators F: $Fu(x)=f(x, u(x))$ for a function $f(x, u)$ of a point x and a real number u; and for non-linear integral operators A: $Au(x)=\int K(x, y, u(y))dy$. *H. Nakano* (Sapporo)

5145:

Brodskiĭ, M. L. On properties of an operator mapping the non-negative part of a space with indefinite metric into itself. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 147–152. (Russian)

Let E be a complex Hilbert space with inner product (x, y), P a p-dimensional projection $(p=1, 2, 3, \cdots)$ on E, $Q=I-P$. Let $[x, y]=(Px, Py)-(Qx, Qy)$ and $\mu(x)=[x, x]$, for all x, $y\in E$. Let $(T, T^+, T^0, T^-)=\{x: x\in E, \mu(x)\ (\geq, >, =, <)\ 0\}$. Theorem: Let A be a linear (not necessarily bounded) operator with domain D_A dense in E such that $A(D_A\cap T)\subset T$. Then R_A has dimension $\leq p$, or the following four properties obtain. 1. A is bounded (and hence we may take $D_A=E$). 2. There is a real number $c>0$ such that $\mu(Ax)\geq c\mu(x)$ if $\mu(x)>0$. 3. A has a p-dimensional invariant subspace $L_1\subset T$ and also an invariant subspace $L_2\subset E\backslash T^+$ such that $\dim E/L_2=p$. 4. There is a real number $r>0$ such that the spectrum of A on L_1 lies in the region $|\lambda|\geq r$ and all other spectral values of A lie in the closed disc $|\lambda|\leq r$. Furthermore, if it is known that $A(T\backslash\{0\})\subset T^+$, then $L_1\subset T^+\cup\{0\}$, $L_2\subset T^-\cup\{0\}$, and $E=L_1+L_2$. For an extended account of a more general situation involving an indefinite metric, see I. S. Iohvidov and M. G. Kreĭn [Trudy Moskov. Mat. Obšč. **5** (1956), 367–432; MR **18**, 320] and the literature there cited. *E. Hewitt* (Seattle, Wash.)

5146:

Èngel'son, Ya. L. On the square root of linear operators in linear topological spaces. Latvijas Valsts Univ. Zinātn. Raksti 8 (1956), no. 2, 73–80. (Russian. Latvian summary)

The results in the present article have been generalized by the author in collaboration with M. M. Vaĭnberg [Dokl. Akad. Nauk. SSSR **122** (1958), 755–758; MR **21** #303]. Let H be a Hilbert space such that $H\supset E$, where E is a real locally convex topological (Hausdorff) linear vector space; E is also required to be reflexive, and H must be an everywhere-dense subset of the dual space E^*. The basic assumption is that $y(x)$ coincides with the inner product (x, y) of H whenever $y\in H$ and $x\in E$. This situation is exemplified by the case $E=L^p(S)$ (where $p>2$ and S is a compact interval). The symbol A is used to denote a completely-continuous mapping of E^* into E whose restriction to H is self-adjoint and quasi-positive [in the sense of Vaĭnberg, Uspehi. Mat. Nauk (N.S.) **10** (1955), no. 4 (66), 189–190]. Suppose further that the eigenvalues of A form a countable set $\{\lambda_k: k=1, \cdots, \infty\}$ such that $0<\lambda_k\leq\lambda_{k+1}$. Let φ_k be the eigenfunction of A that corresponds to λ_k. Consider the operator T defined by the relation $Tu=\sum_{k=1}^{\infty} u(\varphi_k)(\lambda_k)^{-1/2}\varphi_k$ for all u in E^*. The restriction of T to H is the positive square root of A. The object of the paper is to prove the following theorem: the operator T is a linear completely continuous mapping of E^* into H, and its restriction to H is a completely continuous mapping of H into E. This extends a theorem of M. M. Vaĭnberg concerning the case $E=L^p(S)$ [op. cit.]. *G. L. Krabbe* (Lafayette, Ind.)

5147:

Kultze, Rolf. Zur Theorie Fredholmscher Endomorphismen in nuklearen topologischen Vektorräumen. J. Reine Angew. Math. **200** (1958), 112–124.

The author considers a barrelled nuclear space E and a Fredholm operator K in E [in the sense of Grothendieck, Mem. Amer. Math. Soc. no. **16** (1955); MR **17**, 763]. He observes that it is possible to find a closed convex neighborhood V of 0 in E such that the topology of the corresponding quotient space E_V is that of a prehilbert space, and K induces in the Hilbert space $\hat{E}_V$ a Hilbert-Schmidt operator. It is then possible to apply, in particular to the latter, the Smithies formulae [Duke Math. J. **8** (1941), 107–130; MR **3**, 47], and this gives for the solutions of $x-\lambda Kx=y$ formulae of the usual type, provided E is semi-complete. The author also uses his remark to obtain information on the existence of a non-trivial eigenvalue of K. *J. Dieudonné* (Paris)

5148:

★Dubreil-Jacotin, M.-L. Étude algébrique des transformations de Reynolds. Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, pp. 9–27. Centre Belge de Recherches Mathématiques. Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris; 1957. 293 pp. 250 francs belges.

A summary of the work done so far on a class of operators which arose from Osborne Reynolds' attempts [cf. Philos. Trans. Royal Soc. London Ser. A **186** (1895), 123–164] to define averages of functions suited to the study of turbulent flow and compatible with the Navier-Stokes equations.

A Reynolds operator (R.O.) on an algebra A of real functions on a set S is an order-preserving linear transformation of the algebra which satisfies the identity $R(f\cdot Rg+g\cdot Rf)=Rf\cdot Rg+R(Rf\cdot Rg)$. Closely related is an averaging operator (A.O.) [introduced by G. Birkhoff, *Algèbre et théorie des nombres*, Colloques Internat. CNRS no. 24, pp. 143–153, Paris, 1949; MR **13**, 361; and by J. Kampé de Fériet, Sci. Aérienne **3** (1934), 9–34]. This is an order-preserving linear operator which satisfies the identity $A(f\cdot Ag)=Af\cdot Ag$ and such that $Ae=e$, where e is

the identity element of A. Every averaging operator is a Reynolds operator. The work done so far consists in (a) establishing in which algebras every R.O. is an A.O., (b) developing a purely algebraic theory of R.O.'s. The main result re (a) is a proof of a theorem of Molinaro [Publ. Sci. Univ. Alger Sér. A **4** (1957), 87–101; MR **20** #5781] stating that if A is the algebra of real functions on a finite set, then every R.O. is an A.O. An example of Molinaro shows that this is not the case if A is the algebra of bounded continuous functions on the real line. Re (b), the main result is a decomposition theorem into "simple" Reynolds operators, i.e., those for which the only fixed functions are the constants. This is obtained under the additional assumption that the R.O. be regular, i.e., that the set S be partitionable into minimal sets whose characteristic functions are invariant under R.

G.-C. Rota (Cambridge, Mass.)

5149:

Ribeiro Gomes, A.; and Fernandes de Carvalho, J. A. The propagation of error in some operational equations. Rev. Fac. Ci. Univ. Coimbro **27** (1958), 24–28. (Portuguese)

If $A(A+\mu I)x=y$, and if A and y are replaced by $A+\alpha$ and $y+\delta y$, where $\|A^{-1}\|\ \|\alpha\|\leq|\mu|-\|A\|$, then

$$\|\delta x\| \leq \|A^{-1}\|[\|\delta y\|+\|\alpha\|\ \|x\|][|\mu|-\|A\|-\|A^{-1}\|\ \|\alpha\|]^{-1}.$$

A. S. Householder (Oak Ridge, Tenn.)

5150:

Teleman, Silviu. Sur les algèbres de J. von Neumann. Bull. Sci. Math. (2) **82** (1958), 117–126.

Let X be an extremally disconnected (in the sense of the reviewer's thesis [Duke Math. J. **10** (1943), 309–333; MR **5**, 46]—the closure of every open set is open) completely regular space. (A) The space X is extremally disconnected if and only if the lattice of all bounded real-valued continuous functions on X is conditionally complete. (This is a little more than the author proves. For an exhaustive discussion of other properties equivalent to extremal disconnectivity, see Isiwata [Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 147–176; MR **21** #3824].) (B) If every point of X is a G_δ, then X is discrete, as follows at once from (A). (C) If X is compact and the space $C(X)$ of complex-valued continuous functions on X is separable in the uniform topology, then X is finite. If $C(X)$ is separable, then X is imbeddable in a countable product of bounded closed intervals and is thus a separable metric space. Now apply (B).

This result is used to show that if a ring of operators (von Neumann algebra in the terminology of Dixmier [*Les algèbres d'opérateurs dans l'espace hilbertien* (*Algèbres de von Neumann*), Gauthier-Villars, Paris, 1957; MR **20** #1234]) is separable in the uniform topology, then it is finite-dimensional, and is a product of full matrix algebras.

E. Hewitt (Seattle, Wash.)

5151:

Sya, Do-Šin [Shah, Tao-Shing]. On semi-normed rings with involution. Dokl. Akad. Nauk SSSR **124** (1959), 1223–1225. (Russian)

Let R be an algebra with unit e over the complex numbers provided with a family of semi-norms $|x|_n$, where each $|x|_n$ satisfies all of the axioms of a normed algebra except for strict positivity. Suppose that $|x|_n=0$ for all n implies $x=0$. Let R be topologized with the aid of all these semi-norms. Then R is called a semi-normed ring. A number of theorems are announced without proof regarding semi-normed rings, generalizing known facts about normed rings. The following are typical. Let R be complete and suppose that there are only countably many semi-norms on R. Suppose also that R has a continuous involution. Then every positive definite functional on R is continuous. This theorem is false for uncountably many semi-norms, as Michael has shown [Mem. Amer. Math. Soc. no. 11 (1952); MR **14**, 482]. A complete semi-normed ring R with continuous involution has the property that $(e+xx^*)^{-1}$ exists for all $x\in R$ if and only if

$$\sup f(x^*x) = \sup_n \lim_{k\to\infty} (|x^*x|_n)^{1/k},$$

where the sup on the left is taken over all positive definite f such that $f(e)=1$ and the sup on the right is over all seminorms.

E. Hewitt (Seattle, Wash.)

5152:

Perov, A. I. On the principle of the fixed point with two-sided estimates. Dokl. Akad. Nauk SSSR **124** (1959), 756–759. (Russian)

Let H be a real infinite-dimensional Hilbert space and, for $k=0, 1, 2, \cdots$, let $\mathcal{G}_k$ denote the set of all completely continuous self-adjoint linear operators in H which do not have $\lambda=1$ as an eigenvalue and such that the sum of the multiplicities of the eigenvalues $\lambda>1$ equals k. A completely continuous non-linear operator F is defined to have index k in case it has the representation $F(x)=C_x(x)+\phi(x)$, where for each $x\in H$, $C_x\in\mathcal{G}_k$ and $\phi(x)=o(\|x\|)$ as $\|x\|\to\infty$. Theorem 1: Let F have index k and suppose that (i) if $k=0$, then $C_x\leq A_0$, $x\in H$, for some fixed $A_0\in\mathcal{G}_k$, (ii) if $k>0$, then $A_1\leq C_x\leq A_2$, $x\in H$, for fixed $A_i\in\mathcal{G}_k$. Then F has at least one fixed point. Theorem 2: Let F be a completely continuous non-linear operator with a Gâteaux derivative $F'(x)$ such that $x\to F'(x)$ is continuous on H for each fixed h. Suppose that $F'(x)$ belongs to $\mathcal{G}_k$ for all x and that a condition similar to (i) or (ii) with $C_x=F'(x)$ is satisfied. Then F has a unique fixed point in H. For $k=0$ these theorems are known [cf. M. A. Krasnosel'skiĭ, *Topologičeskie metody v teorii nelineĭnyh integral'nyh uravneniĭ*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **20** #3464; and M. M. Vaĭnberg, *Variacionnye metody issledovaniya nelineĭnyh operatorov*, Gosudarstv. Izdat. Tehn.-Teor. Lit, Moscow, 1956; MR **19**, 567]. Applications are given to systems of non-linear integral and differential equations.

R. G. Bartle (Urbana, Ill.)

5153:

Krasnosel'skiĭ, M. A.; and Perov, A. I. Existence of solutions for certain non-linear operator equations. Dokl. Akad. Nauk SSSR **126** (1959), 15–18. (Russian)

Let $T(x)$ be a non-linear operator from a Banach space E_x into a Banach space E_y. Theorem 1: Suppose that the Fréchet derivative $T'(x)$ of $T(x)$ exists and is norm-continuous everywhere. Suppose also that $(Bh)(T'(x)h)\geq \|h\|^2/L(\|x\|)$ $(x, h\in E_x)$, where B is a bounded linear operator from E_x to $E_y{}^*$ with a bounded inverse, and $L(u)$ is a continuous positive function of Osgood type. Then, for any y in E_y, the equation $T(x)=y$ has a unique solution in E_x.

Now let $T(x)$ act in a Hilbert space H. If H_1, H_2 are

closed subspaces of H, with corresponding projections P_1, P_2, we define the "inclination" (rastvor) of H_1 and H_2 to be $\|P_1-P_2\|$. Two bounded self-adjoint linear operators A and B form a proper pair if (i) A^{-1} and B^{-1} are continuous, (ii) the inclination of $H^-(A)$ and $H^-(B)$ is less than 1 ($H^-(A)$ is the subspace of H corresponding to the negative part of the spectrum of A), and (iii) $A\leqq B$. Theorem 2: Let the Gâteaux derivative $T'(x)$ of $T(x)$ be self-adjoint for each x in H; and let $A\leqq T'(x)\leqq B$ ($x\in H$), where A and B are a proper pair of self-adjoint operators. Then, for any y in H, the equation $T(x)=y$ has a unique solution for x.

Next, the authors sketch a simple proof of the theorem of Kneser [S.-B. Preuss. Akad. Wiss. **1923**, 171–174] on the topological connectedness of the system of all solutions of a set of differential equations. From this they obtain, as a generalization of Kneser's theorem, the connectedness of the set of fixed points of certain completely continuous operators on a Banach space. Finally, they take up again the case of a non-linear operator with a continuous Fréchet derivative, carrying a Banach space E_x into a Banach space E_y, and show that, under certain complicated conditions, the existence and uniqueness of solutions of $T(x)=y$ is related to the connectedness properties of E_y-TF, where F is the set of those x for which $T'(x)$ has no inverse.

J. M. G. Fell (Seattle, Wash.)

5154:

Fischer, H. R. Berichtigung zu meiner Arbeit: "Differentialkalkül für nichtmetrische Strukturen. II. Differentialformen". Arch. Math. **10** (1959), 28–30.

The title is self-explanatory. See Arch. Math. **8** (1957), 428–443 [MR **20** #6669].

J. T. Schwartz (New York, N.Y.)

5155:

Grünbaum, B. On a theorem of Kirszbraun. Bull. Res. Council Israel. Sect. **F 7** (1957/58), 129–132.

Let f be a transformation (not necessarily linear) defined on a subset A of a real Banach space X, with values in X. The norm $\|f\|_A$ is defined as the infimum of all a such that $\|f(x_1)-f(x_2)\|\leqq a\|x_1-x_2\|$ for any x_1, $x_2\in A$. The space X has the "extension property" if for each f defined on a subset A of X, with values in a closed convex subset K of X, there exists an extension F to X such that (i) $F(x)=f(x)$ for $x\in A$; (ii) $F(x)\in K$ for $x\in X$; (iii) $\|F\|_X=\|f\|_A$. As noticed by Kirszbraun [Fund. Math. **22** (1934), 77–108], any Euclidean space has the extension property. The author proves, more generally, that X has the extension property if and only if X is an inner-product space or X is a two-dimensional space whose unit sphere is a parallelogram. He remarks that the "only if" part of this theorem becomes false if X is substituted for K in (ii), thus leading to an open question.

L. Nachbin (Rio de Janeiro)

GEOMETRY

See also 4878, 4919, 4920, 4963, 4972, 4981, 5168, 5169.

5156:

Bruins, E. M. The geometry of the plummet. Simon Stevin **33** (1959), 38–60.

An angle-less geometry is developed, using the validity of the Pythagorean theorem $c^2=a^2+b^2$ for the definition of the "plummet line", and tracing the historically warranted "procedure and method" of the Babylonians, preserved in the cuneiform texts IM 55337 and MAH 16055.

Based on 5 obvious assumptions, one of them the Pasch "axiom", another one on the plummet line, a whole series of theorems emerge flawlessly in spite of omitting completely the two essential concepts of logical geometrical structure of the Greeks: angle and parallel. (Here a line m intersecting a line n in a point O is called plummet line in O to n if for any pair of points P on m and Q on n we have $PQ^2=OP^2+OQ^2$). Thus the author proves, e.g., that "Every fourside with sides at three vertices in plummet position has also the sides at the fourth vertex in plummet position; its diagonals are equal", and "In a fourside (a, b, c, d) inscribed to a circle the product of the diagonals $mn=ac+bd$". The theory of the regular n-sides, the problem of doubling the cube, etc., is treated too, observing the historical frame-work of the above limitations.

S. R. Struik (Cambridge, Mass.)

5157:

Steinhaus, H. On triangles. Wiadom. Mat. (2) **1** (1955/56), 169–174. (Polish)

Elementary geometry considers acute-, obtuse- and right-angled triangles, and triangles with two, or all three equal angles, that is, it classifies triangles according to their shape. Therefore, if x, y, z are the sides of a triangle, we may, from that point of view, assume that $x+y+z=1$, without loss of generality. Any triangle is thus determined by its sides x, y, with $z=1-x-y$ (x, y, z, positive), and the point (x, y) in the Cartesian plane may be interpreted as the image of the triangle. The author studies the graphical presentation or map of the triangles he has thus devised. He shows that this map is homeomorphic with a sphere. By varying the radius of that sphere he would be enabled to take care of the size of the triangles as well. Nothing is used beyond the very rudiments of plane and solid analytic geometry, and the discussion is maintained largely on the corresponding level, with perhaps some guarded hints as to other possible ramifications of the subject.

N. A. Court (Norman, Okla.)

5158:

Schopp, J. The inequality of Steensholt for an n-dimensional simplex. Amer. Math. Monthly **66** (1959), 896–897.

Let $A_1, \cdots, A_{n+1}$ be the vertices of the simplex S_n in Euclidean n-space. Let $S_{n-1}^{(i)}$ be the $(n-1)$-face of S_n opposite A_i. Denote its $(n-1)$-dimensional volume by $C_{n-1}^{(i)}$. Let P be an interior point of S_n and let R_i [r_i] be the distance of P from A_i [from $S_{n-1}^{(i)}$]. Then

$$\sum_1^{n+1} C_{n-1}^{(i)}R_i \geqq n\cdot \sum_1^{n+1} C_{n-1}^{(i)}r_i.$$

Cf. Steensholt [same Monthly **63** (1956), 571–572; MR **18**, 146] and Thébault [ibid. **64** (1957), 744–745] for the cases $n=2$ and $n=3$.

P. Scherk (Toronto, Ont.)

5159:

Ewald, Günther. Über den Begriff der Orthogonalität in der Kreisgeometrie. Math. Ann. **131** (1956), 463–469.

In a previous paper [Math. Ann. **131** (1956), 354–371; MR **18**, 502] the author based the geometry of circles in the plane on the concepts of points, circles, incidence of these, and orthogonality. It is shown here that orthogonality can be eliminated (from the axioms) by adding suitable incidence axioms, of which the most interesting is this: (*) If A, B, C, D are 4 distinct points such that there are two circles through C touching each other and touching the circle k through A, B, D at A and B, then the circles through D touching k at A and B also touch each other.

The curves in which the planes intersect a closed convex surface in E^3 satisfy the author's preceding axioms. (*) is necessary and sufficient for these plane sections to be homeomorphic to those of a sphere. This is interesting in view of Reidemeister's attempt to prove Blaschke's conjecture that the only surfaces on which all geodesics passing through a given point meet at a second point are spheres. *H. Busemann* (Los Angeles, Calif.)

5160:

Steinberg, Robert. Finite reflection groups. Trans. Amer. Math. Soc. **91** (1959), 493–504.

The author considers, in Euclidean n-space, a finite set of hyperplanes which is symmetrical by reflection in each one. The reflecting hyperplanes decompose the space into congruent angular regions called chambers, any one of which will serve as a fundamental region for the group G generated by all the reflections. A sufficient set of generators consists of the reflections R_i in the n walls W_i of the fundamental region ($i=1, 2, \cdots, n$). The group G is said to be irreducible if the walls do not fall into two sets, all those in the first set being orthogonal to all those in the second. The products of the n reflections taken in various orders are all conjugate [Coxeter, Ann. of Math. (2) **35** (1934), 588–621; p. 602]; the period of such a product is denoted by h. It is possible to name the n walls in such an order that, for some s, the first s of them are mutually orthogonal, and likewise the remaining $n-s$. This clever trick enables the author to give general proofs for several theorems which had previously been observed by the reviewer and verified laboriously by separate consideration of individual cases. For instance, the total number of reflecting hyperplanes is $nh/2$; and if G contains the central inversion I then h is even and $I=(R_1R_2\cdots R_n)^{h/2}$ [Coxeter, loc. cit., pp. 606, 610].

Letting R_1W_2 denote the image of W_2 by reflection in W_1, and making the conventions $W_k=W_j$ and $R_k=R_j$ if $k\equiv j \pmod n$, the author expresses the $nh/2$ reflecting hyperplanes in the form

$$R_1R_2\cdots R_{k-1}W_k \quad (k=1, 2, \cdots, nh/2).$$

He finds that the reflecting hyperplanes, in this order, contain sets of $n-1$ consecutive vertices of a certain skew nh-gon, namely a "modified Petrie polygon" [Coxeter, *Regular polytopes*, Methuen, London, 1948; MR **10**, 261; pp. 228–231].

The group G is said to be crystallographic if all the dihedral angles between adjacent walls are multiples of either $\pi/4$ or $\pi/6$. In this case it is possible to choose nh vectors $\pm\rho$, orthogonal to the reflecting hyperplanes, of such lengths that the adjunction of the translations 2ρ to G yields an infinite discrete group. Because of their application to the theory of simple Lie groups, these vectors are called roots, and those along inward normals to the n walls W_i, say α_i, are called fundamental roots (Freudenthal's "primary roots"). Every root is expressible in the form $\pm\sum x^i\alpha_i$, where the coefficients x^i are positive integers or zero. In particular, there is a so-called dominant root $\mu=\sum y^i\alpha_i$ whose coefficients are maximal. These vectors α_i and μ are $\frac{1}{2}t_i$ and $\frac{1}{2}z$ in the notation of the reviewer's *Extreme forms* [Canad. J. Math. **3** (1951), 391–441; MR **13**, 443; pp. 404, 410]. The coefficients of α_i in the expression for the dominant root are related to h (and thence to the number of roots) by the simple formula $\sum y^i=h-1$ [*Regular polytopes*, p. 234; *Extreme forms*, p. 413]. *H. S. M. Coxeter* (Toronto, Ont.)

5161:

Manevič, V. A. Polar conjugacy in respect to certain figures of the second order in four-dimensional space. Ukrain. Mat. Ž. **10** (1958), 333–334. (Russian)

Some theorems (in the general case well known) concerning the two types of quadratic cones in four-dimensional projective space are proved by elementary methods. *M. Fiedler* (Prague)

5162:

Lamotke, Klaus. Der Jordansche Polygonsatz in der affinen Geometrie. Enseignement Math. (2) **4** (1958), 272–281.

Jordan's polygon theorem is proven here anew, without utilizing continuity, parallel axiom, Desargues' theorem and congruences. It is deduced from seven axioms, i.e., following Hilbert's numbering of axioms, from I 1 and 2, 3, 4 and II 1, 2, 3, 4. A suggestion by van der Waerden [Christiaan Huygens **13** (1934/35), 65–84] in a paper available only in Dutch, is cited and moreover recent work by W. Graeub utilized. Reference is made to Hans Freudenthal's remark [Nieuw Arch. Wisk. (3) **5** (1957), 105–142; MR **20** #4466] that Hilbert's *Grundlagen* in its 8th (posthumous) edition does not contain a proof of the fundamental Jordan polygon theorem. *S. R. Struik* (Belmont, Mass.)

5163:

★Lombardo-Radice, L. Sur la définition de proposition configurationnelle et sur certaines questions algébro-géométriques dans les plans projectifs. Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, pp. 217–230. Centre Belge de Recherches Mathématiques. Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris; 1957. 293 pp. 250 francs belges.

The definition of "configuration proposition on a free P^n-plane", given by the author [Rend. Sem. Mat. Univ. Padova **24** (1955), 312–345; MR **17**, 776] is compared with that of "Schliessungssatz" investigated by G. Pickert (*Projective Ebenen*, Springer, Berlin-Göttingen-Heidelberg, 1955; MR **17**, 399; n. 1.4]. Certain advantages of the former definition with respect to the latter are pointed out in connection with the question of classifying the propositions of a projective plane, and examples are given. Also there are hints on problems still to be solved in this direction. *B. Segre* (Rome)

5164:

★Segre, Beniamino. Sulle geometrie proiettive finite.

Convegno internazionale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 46–61. Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

In a projective plane a k-arc is a set of k points no three on a line. A k-arc is called complete if it is not part of a $(k+1)$-arc, otherwise incomplete. In a plane of order q, the maximum possible value for k in a k-arc is $q+1$ if q is odd and $q+2$ if q is even. A k-arc for $k=q+1$ or $q+2$ is called an oval. This paper treats k-arcs in Desarguesian finite planes. For q odd an oval is necessarily an irreducible conic. For $q=2^h$ the tangents to an irreducible conic meet in a point, the nucleus of the conic, and this point adjoined to the conic gives an oval of $q+2$ points. For $q=2^h$ and $h=5$ or $h\geqq 7$ there are further types of ovals, given by equations $y=x^r$ with $r=2^g$ for an appropriate integer g, $2\leqq g\leqq h-2$. There are a number of results on completion of k-arcs. Thus for odd q, with $q\geqq 16r^2+65r+29$, every $(q-r)$-arc can be completed in a unique way to an oval. For $q=8$ there are complete 6-arcs.

Marshall Hall, Jr. (Pasadena, Calif.)

5165:

Zappa, Guido. Piani grafici. Rend. Sem. Mat. Fis. Milano **28** (1959), 78–86. (English summary)

This is a brief discussion of some known results on Desarguesian and non-Desarguesian planes. It includes the conjecture that every finite non-Desarguesian plane contains a Fano subplane.

Marshall Hall, Jr. (Pasadena, Calif.)

5166:

Al-Dhahir, M. W.; and Shekoury, R. N. Constructions in the hyperbolic plane. Proc. Iraqi Sci. Soc. **2** (1958), 1–6. (Arabic summary)

The following problems of hyperbolic non-Euclidean geometry (analogous with those of Euclidean geometry) are solved by construction with compass and straight edge, using the theory of corresponding points and the theory of associated triangles [see H. E. Wolfe, *Introduction to non-Euclidean geometry*, Dryden, New York, 1945]: Construction of tangent lines to a given horocycle parallel [resp. perpendicular] to a given line; construction of the internal and external tangent lines to two given circles [resp. horocycles]; and construction of a horocycle, tangent to two given horocycles.

S. R. Struik (Cambridge, Mass.)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 5140.

5167:

Berge, Claude. Sur une propriété combinatoire des ensembles convexes. C. R. Acad. Sci. Paris **248** (1959), 2698–2699.

L'A. démontre, dans un espace vectoriel topologique ordinaire (c'est à dire, dans tout espace vectoriel topologique qui induit sur une variété linéaire de dimension n la topologie de R^n) le théorème suivant. Soit, dans un ensemble convexe compact non vide X, une famille (finie ou non) d'ensembles convexes compacts C_i et deux entiers m et n tels que: (1) l'intersection de n quelconques des C_i est non vide; et (2) la réunion de m quelconques des C_i recouvre X. Si $m\leqq n+1$, alors l'intersection de toute la famille est non vide. La démonstration s'appuie sur le théorème de Helly et le lemme de Sperner.

L. A. Santaló (Buenos Aires)

5168:

Molnár, I. Problèmes non résolus de géometrie. Gaz. Mat. Fiz. Ser. A **10** (**63**) (1958), 730–735. (Romanian. French and Russian summaries)

The author discusses four unsolved problems such as: Find the maximum radius of a sphere which can be covered by four spheres whose radii are given.

H. S. M. Coxeter (Toronto, Ont.)

5169:

Fejes Tóth, L. Kugelunterdeckungen und Kugelüberdeckungen in Räumen konstanter Krümmung. Arch. Math. **10** (1959), 307–313.

The author summarizes some of his earlier work on packing and covering, fills in some of the gaps in a paper by the reviewer [Acta Math. Acad. Sci. Hungar. **5** (1954), 263–274; MR **17**, 523], and gives a beautiful drawing of the regular hyperbolic tessellation $\{\infty, 3\}$, whose faces are the Dirichlet regions for the closest packing and thinnest covering of horocycles in the hyperbolic plane. Turning to Euclidean space, he conjectures that the density of a packing of equal spheres is $\leqq 0.799\cdots$ and that the density of a covering by equal spheres is "$\leqq 1.428\cdots$". (He obviously intended to write "$\geqq 1.432\cdots$".) Meanwhile the former conjecture has been established by C. A. Rogers [Proc. London Math. Soc. (3) **8** (1958), 609–620; MR **21** #847], and the latter by the reviewer in collaboration with L. Few and the same Rogers [Mathematika **6** (1959), 147–157].

H. S. M. Coxeter (Toronto, Ont.)

5170:

Halberg, Charles J. A., Jr.; Levin, Eugene; and Straus, E. G. On contiguous congruent sets in Euclidean space. Proc. Amer. Math. Soc. **10** (1959), 335–344.

Theorem: Let S be the closed interior of a Jordan curve in E_2. Then for any direction θ there exist six translates of S with the following properties. (1) One translate is in the direction θ. (2) Each translate is contiguous to S (i.e., their intersection is non-empty and is equal to the intersection of their frontiers). (3) No two translates have interior points in common. (4) Each translate is contiguous to two others. (5) The union of the six translates encloses S. For the generalization to E_n one considers reflected sets S_i with respect to hyperplanes of support of S; then, there exist a number c_n of sets S_i whose convex hulls have no interior points in common; the number c_n is known only for $n=1, 2, 3$ ($c_1=2, c_2=6, c_3=12$). The authors give the following asymptotic evaluation

$$\log(2\cdot 3^{1/2}/3) \leqq \liminf \frac{\log c_n}{n} \leqq \limsup \frac{\log c_n}{n} \leqq \log 2.$$

[Related topics were considered by Hadwiger, Arch. Math. **8** (1957), 212–213; MR **19**, 977.]

L. A. Santaló (Buenos Aires)

GENERAL TOPOLOGY, POINT SET THEORY

See also 5190, 5226.

5171:

Straus, E. G. On a problem of W. Sierpiński on the congruence of sets. Fund. Math. **44** (1957), 75–81.

Le but de cette Note est de résoudre complètement un problème de M. W. Sierpiński [Fund. Math. **37** (1950), 1–4; Elem. Math. **5** (1950), 1–4; MR **12**, 597, 809] sur l'existence d'un ensemble de points qui est congruent à ceux obtenus en négligeant un point singulier dans celui-ci. D'après un résultat de M. W. Sierpiński [Fund. Math. loc. cit.], tout ensemble S dans l'espace euclidien E_1 de dimension 1 peut contenir au plus un point p tel que $S-\{p\}\cong S$, mais d'après cette Note, cette proposition est aussi vraie pour les ensembles S dans l'espace euclidien E_2 de 2 dimensions. Pour le voir, l'auteur suppose l'existence d'un ensemble S dans E_2 et deux points p et q de S tels que $\varphi S=S-\{p\}$ et $\psi S=S-\{q\}$, où φ et ψ sont isométriques. Alors, au moins un de φ et ψ est une rotation, et si c'est φ, on a

$$\varphi^{-m} = \psi^{-n}\varphi^{-2m}\psi^{n}\varphi^{m}\psi^{-n}\varphi^{m}\psi^{n}\varphi^{-2m}\psi^{-n}\varphi^{m}\psi^{n}.$$

Dès lors, on peut choisir les nombres naturels m et n de manière que $p_0=p,\ p_1=\psi^n p_0, \cdots, p_{11}=\psi^{-n}p_{10}=\varphi^{-m}p$ appartiennent à S. C'est contradictoire.

Or, dans l'espace euclidien E_3 de 3 dimensions, on peut trouver un ensemble M de points ainsi que pour chaque point x de M, il existe une rotation φ tel que $\varphi M=M-\{x\}$. Ce résultat est appuyé par le théorème suivant. Soient G un groupe libre de rang $\geqq 2$ et $\mathfrak{m}$ un nombre cardinal tel que card $G^{\mathfrak{m}}=$ card G. Alors, il existe un sous-ensemble U de G ayant card $U=$ card G et tel que, pour chaque sous-ensemble Q de U ayant card $Q\leqq\mathfrak{m}$, il existe un élément φ_Q de G tel que $\varphi_Q U=U-Q$. Ce théorème est discuté aussi en concernant la théorie des groupes.

M. Kondô (Tokyo)

5172:

Straus, E. G. On Sierpiński sets in groups. Fund. Math. **45** (1958), 332–333.

D'après la définition de l'auteur, un ensemble de Sierpiński est celui qui est congruent à chaque sous-ensemble propre et maximal; quand un groupe contient un sous-groupe libre de rang 2, il contient un ensemble de Sierpiński [#5171]. Or, dans cette Note, l'auteur a démontré le théorème suivant: quand un groupe contient un ensemble de Sierpiński, il contient aussi un sous-groupe libre de rang 2.

M. Kondô (Tokyo)

5173:

Fort, M. K., Jr. Images of plane continua. Amer. J. Math. **81** (1959), 541–546.

It is shown that the plane continuum K consisting of two concentric circles together with a spiral between them approaching each circle asymptotically cannot be the continuous image of any plane continuum which fails to separate the plane. Also it is proved that no plane continuum can be mapped continuously onto the dyadic solenoid. These results are related to the general question of existence of a universal domain continuum for the class of all continua and to the similar question for plane continua. It thus follows that no plane continuum is such a universal domain for all continua and, likewise, a plane continuum not separating the plane cannot be such a universal domain for plane continua.

G. T. Whyburn (Charlottesville, Va.)

5174:

Doyle, Patrick H. A wild triod in three-space. Duke Math. J. **26** (1959), 263–267.

{Editor's note: The following summarizes a communication from the author.} The author constructs a triod T in 3-space S, i.e., a homeomorphic image of three line segments with only one common point which is an end-point of each. The proof that T is wild is not correct since the relations defining the group of $S\backslash T$ are incorrect. Furthermore the corrected relations fail to yield the result since the group has local properties which are precisely those of a tame set. A complete discussion of this matter will be found in a forthcoming paper in Duke Math. J. by R. H. Fox and H. Debrunner. These authors do use Lemma 1 of the present paper, and their sets are similar to T above, though they are shown to be wild.

5175:

Hönig, Chaim Samuel. Sur quelques propriétés topologiques des P-espaces. An. Acad. Brasil. Ci. **29** (1957), 319–322. (Portuguese summary)

A P-space is a completely regular space X such that the intersection of any countable family of open sets is open, or equivalently, such that in the ring of all continuous real-valued functions on X every prime ideal is maximal [Gillman and Henriksen, Trans. Amer. Math. Soc. **77** (1954), 340–362; MR **16**, 156]. Let R_d be the real line with the discrete topology and for any index set I let $(R_d{}^I)^*$ be the topological space whose underlying set is the Cartesian product of real lines indexed by I, and whose open sets have as base all sets of form ΠV_i where $V_i=R_d$ for all but countably many indices i (and each V_i is open in R_d). The author proves that a space is a P-space if and only if it is homeomorphic to a subset of $(R_d{}^I)^*$ for some I. Let X be a P-space. The author calls the weakest uniform structure on X for which each continuous function from X to R_d is uniformly continuous (R_d equipped with the discrete uniform structure) the natural uniform structure on X. The natural uniform structure on X is compatible with the topology of X, and it is complete if and only if X is a Q-space in the terminology of Hewitt [ibid. **64** (1948), 45–99; MR **10**, 126] (realcompact in the terminology of Gillman and Jerison's *Rings of continuous functions* [Van Nostrand, Princeton, N.J.; in print]).

S. Warner (Princeton, N.J.)

5176:

Arhangel'skiĭ, A. An addition theorem for the weight of sets lying in bicompacts. Dokl. Akad. Nauk SSSR **126** (1959), 239–241. (Russian)

Answering a question of Alexandroff-Urysohn, Yu. M. Smirnov [Fund. Math. **43** (1956), 387–393; MR **18**, 813] proved that the weight of any locally compact space which is the union of $\leqq\aleph_0$ subspaces, each of a weight $\leqq\aleph_0$, is also $\leqq\aleph_0$. In the present paper the preceding result is generalized in such a way that any cardinality τ (and not only $\tau=\aleph_0$) is considered. The main concept is one of a net: any family H of sets such that for each point

x of a space S and each neighborhood O_x of x there exists a $\Gamma_x \in H$ such that $x \in \Gamma_x \subseteq O_x$ is called a net of the space S. The connection between nets and bases of S is given by the following theorems 1 and 3 respectively: In every bicompact [locally bicompact] space possessing a net of a cardinality $\leq \tau$ there exists a basis of a cardinality $\leq \tau$. By means of these theorems one proves: Let S be a space that (locally) is initially compact up to cardinality τ; if S is the union of $\leq \tau$ subspaces, each with a net of cardinality $\leq \tau$, then S is a (locally) bicompact space of a weight $\leq \tau$ (Th. 5). Using nets, some results of P. S. Alexandroff are proved, in particular that the weight of a bicompact B does not increase when B is continuously transformed onto a bicompact.

Đ. Kurepa (Princeton, N.J.)

5177:

Parovičenko, I. I. Anti-Urysohn extensions of topological spaces. Dokl. Akad. Nauk SSSR **126** (1959), 280–283. (Russian)

Let m be a cardinal number. A topological space R is of class $\sim T_u(m)$ if the intersection of less than m closed nonvoid neighborhoods is never void. $\sim T_u(3)$-spaces are also called anti-Urysohn spaces. Urysohn, Pospíšil, and Bing have each exhibited anti-Urysohn spaces. In this paper is established a condition sufficient for a space R to be embedded in a $\sim T_u(\aleph_0)$-space S. This condition is satisfied by each infinite-dimensional Banach space, the rationals, and some finite-dimensional separable metric connected spaces. The condition involves being "anti-locally compact", i.e., no point has a compact neighborhood.

R. Arens (Los Angeles, Calif.)

5178:

Smirnov, Yu. M. Universal spaces for certain classes of infinite-dimensional spaces. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 185–196. (Russian)

The author proves the existence of a universal space for infinite-dimensional (in the weakest sense) spaces with a countable basis (for spaces decomposable into the sum of a countable number of closed finite-dimensional sets) and constructs an example of a compactum which is decomposable into the sum of a countable number of zero-dimensional sets but is not infinite-dimensional in the weakest sense of the word. Also he introduces the large transfinite dimension Ind.

Author's summary

5179:

Sklyarenko, E. G. Dimensionality properties of infinite-dimensional spaces. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 197–212. (Russian)

If A, B are subsets of a topological space R, a partition between A and B is a closed subset C of R such that $R-C$ is a union of disjoint open sets G, H, with $A \subset G$, $B \subset H$. A space R is weakly infinite-dimensional (wid) if for every countable family of pairs of closed sets A_i, B_i, $A_i \cap B_i = \emptyset$, there exists a sequence of partitions C_i (with C_i a partition between A_i and B_i) such that (1) only a finite number of the C_i are distinct, and (2) $\bigcap_i C_i = \emptyset$. In the contrary case, R is said to be strongly infinite-dimensional (sid). This definition is due to Smirnov, and for compact spaces is equivalent to Aleksandrov's definition [the latter does not require Condition (1)]. An example (the subset of the Hilbert parallelotope consisting of all points with only a finite number of nonvanishing coordinates) shows that this definition is not the same as Aleksandrov's (or most others). In the sequel all spaces are assumed to be normal.

The main results of the paper are the following. (i) Every non-compact wid space with a countable basis can be topologically imbedded in a wid compactum (compact metric space). (ii) A non-compact wid space is characterized by the fact that it can be written as a union of a wid compactum and a countable number of finite-dimensional open sets which satisfy a certain "convergence" condition. (iii) Every sid bicompactum (compact space) contains an infinite-dimensional Cantorian manifold (author's definition: an infinite-dimensional Cantorian manifold is a bicompactum which is not separated by any wid closed set). (iv) If R is normal, dim $R \leq n$ (finite) if and only if an arbitrary countable family of pairs of disjoint closed sets can be separated by $n+1$ partitions.

The author notes that the results cannot be formally generalized to noncountable families of pairs of disjoint closed sets.

H. Komm (Troy, N.Y.)

5180:

Rainwater, John. Spaces whose finest uniformity is metric. Pacific J. Math. **9** (1959), 567–570.

The author adds some new characterizations, of the spaces described in the title, to the known ones [J. Nagata, J. Inst. Polytech. Osaka City Univ. **1** (1950), 28–38; MR **12**, 272; B. T. Levšenko, Mat. Sb. (N.S.) **42** (84) (1957), 479–484; MR **20** #2679; M. Atsuji, Pacific J. Math. 8 (1958), 11–16; MR **20** #5468]. The following results are typical. For a metrizable space S to be such that its finest (compatible) uniformity is metrizable, each of the following is necessary and sufficient: every subset (or closed subset) of S has a compact boundary; every closed continuous image of S is metrizable; every Hausdorff quotient space of S is metrizable (or first countable). The uniformities in question are characterized by the property that every closed discrete subspace of S is uniformly discrete. {The incidental remark on p. 570, l. 1 seems to the reviewer to be incorrect; in the example referred to, a symmetrical Cantor set would be non-metrizable, though the boundary of a closed set.}

A. H. Stone (Manchester)

5181:

McDougle, Paul. Mapping and space relations. Proc. Amer. Math. Soc. **10** (1959), 320–323.

A space X is called an "M space" if no sequence of points of X can converge to two distinct points of X; X is an "E space" if, for every $A \subset X$ and $b \in \bar{A}$ there is a sequence of points of A converging to b. [For other terminology, and background, see the author's paper in Proc. Amer. Math. Soc. **9** (1958), 474–477; MR **20** #1971.] Suppose throughout that f is a continuous mapping of an M, E space X onto a space Y. If f is quasicompact, then Y is an M space if and only if f is semi-closed; and Y is an M, E space if and only if f is semi-closed and P_1. If X is metrisable, and f is either closed or "compact" (i.e., $f^{-1}(C)$ is compact for each compact $C \subset Y$), then Y is metrisable if and only if f is P_2. The results are applied to certain types of decomposition spaces (introduced by W. E. Malbon in an unpublished dissertation), and illustrated by an example.

A. H. Stone (Manchester)

ALGEBRAIC TOPOLOGY

See also 5226.

5182:

Wu, Wen-tsün. On the relations between Smith operations and Steenrod powers. Fund. Math. **44** (1957), 262–269.

By using the theory of P. A. Smith and M. Richardson [Ann. of Math. (2) **39** (1938), 611–633] we can introduce a system of homomorphisms

$$\mathrm{Sm}_k^{(p)}: H_r(K, I_p) \to H_{r-k}(K, I_p).$$

On the other hand, we have the Steenrod reduced powers

$$\mathrm{St}_{(p)}^k: H^r(K, I_p) \to H^{r+k}(K, I_p).$$

The author previously found [Colloque de Topologie de Strasbourg, 1951, no. IX, Bibliothèque Nat. Univ. Strasbourg, 1952; MR **14**, 491] that these two systems of homomorphisms are equivalent in the sense that one determines the other. In this paper he discusses the mode of their mutual determination explicitly without using Thom's axiomatic theory of Steenrod powers [ibid., no. VII, MR **14**, 491].

References: M. Nakaoka, Sûgaku **8** (1956/57), 72–83; Seminar report of topology, Osaka City University, 1955.

H. Uehara (Los Angeles, Calif.)

5183:

Wu, Wen-tsün. On the reduced products and the reduced cyclic products of a space. Jber. Deutsch. Math. Verein. **61** (1958), Abt. 1, 65–75.

Let X be an arbitrary space, $\tilde{X}_p$ the p-fold topological product of X with itself, $\tilde{\Delta}_p$ the diagonal in $\tilde{X}_p$, and t the cyclic transformation in $\tilde{X}_p$. The space $X_p = \tilde{X}_p/t$ is called the p-fold cyclic product of X. Denote the natural projection of $\tilde{X}_p$ onto X_p by π. The image $\Delta_p = \pi(\tilde{\Delta}_p)$ will be called the diagonal in X_p. The spaces obtained from $\tilde{X}_p$ and X_p by removing the diagonals are called the p-fold reduced product and the p-fold reduced cyclic product of X, respectively. As applications of these reduced products, the author summarizes in this paper a number of his recent contributions, mostly published in Chinese [Acta Math. Sinica **3** (1953), 261–290; **5** (1955), 505–552; **7** (1957), 79–101; **8** (1958), 79–94; Bull. Acad. Polon. Sci. Cl. III **4** (1956), 573–577; MR **17**, 290, 883; **20** #3536, #4825; **18**, 664; and #5182 above.]

Sze-tsen Hu (Detroit, Mich.)

5184:

Yo, Ging-tzung. Homology operations. I, II. Sci. Record (N.S.) **2** (1958), 332–337, 426–434.

W. T. Wu [#5182 above] has defined homology operations

$$\mathrm{Sm}_i^{(p)}: H_q(K; Z_p) \to H_{q-i}(K; Z_p)$$

for every finite simplicial complex K, using the Smith theory on periodical transformations. In these two papers the author generalizes these operations to

$$\mathscr{S}_i^{(p)}: H_q(K, L; G) \to H_{q-i}(K, L; G'_{(i)}),$$

where K is a regular cell complex, G is arbitrary, and $G'_{(i)} = G \otimes \cdots \otimes G$ (p factors) if i is odd, $= G \otimes \cdots \otimes G \otimes Z_p$ if i is even.

The author also establishes the basic properties of $\mathscr{S}_i^{(p)}$; e.g., if L is a subcomplex of K, $\partial: H_q(K, L; G) \to H_{q-1}(L; G)$ the boundary homomorphism, and $x \in H_q(K, L; G)$, then

$$\mathscr{S}^{(p)}_{2i+1}\partial x = (-1)^{((p-1)/2)i}\left(\frac{p-1}{2}\right)!\,\partial\mathscr{S}^{(p)}_{2i+1}x \quad (p > 2),$$

$$\mathscr{S}_i^{(2)}\partial x = \partial\mathscr{S}_i^{(2)}x.$$

F. P. Peterson (Cambridge, Mass.)

5185:

Yo, Ging-tzung. The Smith algebra. Sci. Record (N.S.) **3** (1959), 185–191.

The author continues his study [review above] of operations

$$\mathrm{Sm}_k^{(p)}: H_q(K, L; Z_p) \to H_{q-k}(K, L; Z_p).$$

If we denote the dual cohomology operations by

$$\mathrm{Sm}_{(p)}^k: H^q(K, L; Z_p) \to H^{q+k}(K, L; Z_p),$$

the author shows that these satisfy certain axioms for Steenrod powers, but not the statement $\mathrm{Sq}^k(\alpha) = \alpha^2$ if $k = \dim \alpha$ and $p = 2$. Hence this axiom is independent of the others. {Reviewer's note: the operations $\mathrm{Sq}^0 =$ identity and $\mathrm{Sq}^k = 0$ for $k > 0$ trivially show this fact.}

Let $C: A^* \to A^*$ be the conjugation in the Steenrod algebra [Milnor, Ann. of Math. (2) **67** (1958), 150–171; MR **20** #6092]. Then the author shows that $C(\mathrm{Sq}^k) = \mathrm{Sm}_{(2)}^k$ and the generalization for odd primes.

F. P. Peterson (Cambridge, Mass.)

5186:

★**Hu, Sze-Tsen. Homotopy theory.** Pure and Applied Mathematics, Vol. VIII. Academic Press, New York-London, 1959. xiii + 347 pp. $11.00.

This book is a most welcome addition to the literature of algebraic topology. A difficulty hitherto facing anybody who wished to lecture or to research in homotopy theory has been the need to go to many different sources for the basic material of the subject. Moreover, the sources in question have not (with the exception of a brief monograph by the reviewer [*An introduction to homotopy theory*, Cambridge Univ. Press, 1953; MR **15**, 52]) been written for the benefit of the inexperienced and have (without exception!) been unsuitable for beginners.

The book under review is comprehensive and satisfying and deserves to become a standard reference work. The author assumes only familiarity with singular homology and proceeds to describe in fair detail the main ideas of algebraic homotopy theory. Chapter 3 deals with fibre spaces, Chapters 4 and 5 with homotopy groups and Chapter 6 with obstruction theory. Chapter 7 deals with cohomotopy groups, but the reviewer would have preferred a chapter on S-theory, which appears to be a much more important topic.

The last four chapters, probably the most useful to practising topologists, treat spectral homology theory. Chapter 8 contains Massey's algebraic theory of spectral sequences derived from exact couples and describes the exact couple associated with a filtered differential group. This is applied in the next chapter to the fibre space situation, the treatment following very closely the lines of Serre's thesis. Serre's C-theory is studied in Chapter 10 and applied in the final chapter to the calculation of the homotopy groups of spheres.

The book is not strictly self-contained in that much of

the later material depends on results and notions drawn from exercises to earlier chapters; exercises, moreover, which no reader is expected to do. The author admits that, to tackle the exercises, the reader is usually expected to read the indicated papers; the latter, in fact, usually solve the exercises completely. The expedient of describing germane material as an exercise appears dubious and it is particularly disconcerting that the exercises range in difficulty from the trivial to so Herculean a labour as that of defining the Steenrod squares and verifying the Thom axioms. Indeed the role of the extra structure in cohomology in homotopy calculations requires more space than it is here given: multiplicative structure appears only among the exercises to Chapter 8 and there is no proof that the multiplicative structure defined on the cubical cochains satisfies the axioms. However, we certainly do not criticize the author for leaving some hard work to the reader, nor for expecting the reader to supplement his reading of the text by the study of certain well-chosen original papers.

There are some errors and obscurities. For example, the reference given on p. 57 to a generalized Hurewicz theorem refers to a non-existent section. Again, the proof on p. 299 that C-equivalence is an equivalence relation appears to be incorrect (and the statement of Proposition 4.1 incomplete). The fuss about Lemma 2.2 on p. 312 is difficult to understand since the assertion is trivial using reduced suspension and the Hurewicz theorem. The author's definition of a c.s.s. complex conflicts with current usage—this should be explicit.

Material that the reviewer would have included (e.g., combinatorial homotopy theory, semi-simplicial homotopy theory, Postnikov decompositions) has been omitted. But tastes must vary, and, in the main, the author has shown very good judgment in the selection of material. Certainly he has placed us all in his debt by providing a clear exposition to which we can confidently refer in selling homotopy theory to students and colleagues.

P. J. Hilton (Birmingham)

5187:

Aleksandrov, P.; and Pasynkov, B. Elementary proof of the essentiality of the identical mapping of a simplex. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 5 (77), 175–179. (Russian)

This brief note contains two elementary proofs of the basic fact that there is no continuous map of a simplex onto its boundary, leaving the boundary fixed.

D. W. Kahn (Cambridge, Mass.)

5188:

Hadwiger, H. Elementare Begründung ausgewählter stetigkeitsgeometrischer Sätze für Kreis und Kugelfläche. Elem. Math. **14** (1959), 49–60.

This paper contains strictly elementary and very simple proofs for a number of basic topological results specialized to the two-dimensional case. Among these are the Brouwer fixed-point theorem and the Poincaré result that no continuous tangential vector field exists on the sphere S_2. Some of the results in the two-dimensional form are stronger than the specializations of the corresponding n-dimensional results. A proof is given that if a sphere with centre O is covered by three closed sets, then if ρ is a number with $0<\rho\leqq\pi$ at least one of the covering sets contains two points P_1, P_2 so that the angle P_1OP_2 has magnitude ρ. For $\rho=\pi$ this is a specialization of the result that if a sphere S_n is covered by $n+1$ closed sets at least one of the sets contains a pair of antipodal points. Extensive references are given. *D. Derry* (Vancouver, B.C.)

5189:

Eckmann, Beno. Groupes d'homotopie et dualité. Bull. Soc. Math. France **86** (1958), 271–281.

In this article the author, summarizing the results of joint work with the reviewer, indicates how the main concepts of algebraic topology may be developed from the basic notion of the set $\Pi(A, B)$, where A, B are objects in the category $\mathcal{T}$ of based spaces and Π is the set of based homotopy classes of maps from A to B. The sets $\Pi(A, B)$, A variable, B fixed, admit a natural group structure if and only if B is a group (up to homotopy) in $\mathcal{T}$ and, dually, the sets $\Pi(A, B)$, A fixed, B variable, admit a natural group structure if and only if A is a 'cogroup' (up to homotopy) in $\mathcal{T}$. Moreover if A is a cogroup and B a group the induced group structures in $\Pi(A, B)$ coincide and are abelian. Among the groups are the loop spaces and among the cogroups are the suspensions; in the former case the group structure in Π generalizes the cohomology groups of A and, in the latter case, the homotopy groups of B. The basic notion is relativized and dual exact sequences defined, generalizing the familiar exact sequences of algebraic topology. Homotopy groups with coefficients are introduced (they have also been studied by Peterson and others) and a universal coefficient theorem obtained as a consequence of the exact sequences. Finally the author points to certain further aspects of the duality; the topics he instances (and others) have all been treated in joint publications of the author and the reviewer. *P. J. Hilton* (Birmingham)

5190:

McCandless, Byron H. Test spaces for metric spaces. Proc. Amer. Math. Soc. **10** (1959), 372–376.

Let $\mathcal{T}$ be some (full) subcategory of the category of topological spaces; following Kuratowski we write $X\tau Y$ if $X, Y\in\mathcal{T}$ and, for any closed subset C of X and any map $f: C\to Y$, there exists an extension of f to the whole of X. The author describes Y as a test space for dimension n if $X\tau Y$ is true when and only when $\dim X\leqq n$. In an earlier paper [same Proc. **7** (1956), 1126–1130; MR **18**, 662] the author took $\mathcal{T}$ to be the category of separable metric spaces and Y to be compact, n-dimensional and n-LC (and 1-simple if $n=1$) and then obtained necessary and sufficient conditions for Y to be a test space for dimension n. In the present paper the author takes $\mathcal{T}$ to be the category of metric spaces and drops the requirement that Y be compact, and obtains precisely the same necessary and sufficient conditions. The added generality is obtained by using the 'classical' theorem of J. H. C. Whitehead to show that a space Y satisfying the authors' conditions (3^n, 4^n, 5^n in the review cited above) must have the homotopy type of a bunch of n-spheres.

P. J. Hilton (Birmingham)

5191:

Bucur, Ion. Sur une extension d'un théorème de Wu. Rev. Math. Pures Appl. **3** (1958), 427–429.

In his paper, Ann. of Math. (2) **69** (1959), 414–420

[MR 21 #1593], Peterson proved that a $U(n)$ bundle over a complex K of dimension $\leq 2n$, satisfying the condition that any torsion in $H^{2j}(K)$ is relatively prime to $(j-1)!$, is trivial if and only if all its Chern classes are zero. In particular this holds for a $U(3)$-bundle over a 6-dimensional complex with no 2-torsion in $H^6(K)$. The author gives a different proof for this latter result with the stronger assumption that there is no torsion at all in $H^6(K)$. The strong assumption seems to be essential for this proof.

N. Stein (New York, N.Y.)

5192:

de Carvalho, Carlos A. A. Classes de Smith associées à un espace fibré. Classes caractéristiques. C. R. Acad. Sci. Paris **247** (1958), 1947–1950.

Soient E un espace fibré de base B et fibre F et p un nombre premier. L'auteur désigne par $\bar{E}^p$ la restriction du fibré produit E^p à la diagonale B de B^p, et par $\bar{E}_*{}^p$ le fibré associé dont la fibre est le produit F^p privé de sa diagonale; c'est un revêtement à p feuillets du quotient $\bar{E}_*{}^{[p]}$ de $\bar{E}_*{}^p$ par l'action du groupe cyclique d'ordre p opérant par permutation cyclique des facteurs.

L'auteur considère le cas où le p-produit cyclique de F privé de sa diagonale a la même homologie mod. p que si F était un espace numérique. Utilisant les classes de Smith du revêtement $\bar{E}_*{}^p \to \bar{E}_*{}^{[p]}$ (autrement dit les classes caractéristiques de ce revêtement), il retrouve une définition, indépendante du groupe de structure, des classes de Stiefel-Whitney de E $(p=2)$ et des classes caractéristiques de Wu $(p>2$, E fibré vectoriel orientable).

A. Haefliger (Princeton, N.J.)

5193:

de Carvalho, Carlos A. A. Classes de Smith. Existence des sections. C. R. Acad. Sci. Paris **247** (1958), 2081–2083.

L'auteur considère cette fois la restriction $\bar{E}^p$ de E^p au complémentaire de la diagonale dans B^p [cf. note précédente]. Il obtient une condition nécessaire pour l'existence d'une section de E en considérant les classes de Smith des revêtements à p feuillets définis par l'action du groupe des permutations cycliques opérant sur $\bar{E}^p$ et sur sa base.

A. Haefliger (Princeton, N.J.)

5194:

de Carvalho, Carlos A. A. Sur le plongement des espaces fibrés. C. R. Acad. Sci. Paris **247** (1958), 2268–2270.

Un plongement d'un espace fibré (E_1, φ_1, B_1) dans un espace fibré (E_2, φ_2, B_2) est une application continue f de E_1 dans E_2 dont la restriction à chaque fibre de E_1 est une application biunivoque de cette fibre sur un sous-espace fermé d'une fibre de E_2.

Les fibres de E_1 et E_2 vérifiant les mêmes hypothèses que dans la première note, l'auteur donne des conditions nécessaires pour l'existence d'un plongement de E_1 dans E_2 en utilisant les relations entre classes caractéristiques et classes de Smith démontrées précédemment. (Ces conditions semblent équivalentes à celles qu'on obtient en exprimant que le fibré induit de E_2 par une application continue de B_1 dans B_2 est somme de Whitney de E_1 et d'un fibré complémentaire.)

A. Haefliger (Princeton, N.J.)

5195:

Norguet, François. Sur la théorie des résidus. C. R. Acad. Sci. Paris **248** (1959), 2057–2059.

Let S be a closed subset of a locally compact X such that S and $X-S$ are paracompact manifolds of dimensions m and n, respectively. Let H and $\bar{H}$ denote homology [or cohomology] with closed support and compact support, respectively. If

$$\partial: \bar{H}^p(S) \to \bar{H}^{p+1}(X-S)$$

and

$$\delta: \bar{H}_p(S) \to \bar{H}_{p+n-m-1}(X-S),$$

isomorphically, then the transpose r of δ $(r: H^p(X-S) \to H^{p-n+m+1}(S))$ relative to the duality between $\bar{H}$ and H is called the residue. If X and S are of class C^∞, with S regularly imbedded in X, and if B is the boundary of a certain closed neighborhood of S such that B has class C^∞ and is fibred by spheres, then r is the same as $\theta: H^p(B) \to H^{p-n+m+1}(S)$. If X and S are complex analytic, and S, regularly imbedded, has codimension 1, then r is closely related to a certain cohomology class defined by Leray. For a finite decreasing sequence $\{X_i\}$ of closed subsets of X such that $X_i - X_{i+1}$ is a manifold, spectral residues are defined. Finally, for two closed subsets of X a composition of residues is defined which is an alternating operation.

F. D. Quigley (New Orleans, La.)

5196:

Poenaru, Valentin. Considérations sur les variétés simplement connexes à 3 dimensions. Rev. Math. Pures Appl. **3** (1958), 139–156.

This paper summarizes various results, with underlying definitions and analysis, to appear in detail elsewhere. Each of the three theorems relates to a triangulation K_3 of a 3-manifold and to representations of K_3 based on a "ψ-representation" $(F_\psi(K_3), \Phi)$ defined as follows. (1) $F_\psi(K_3)$ is an everywhere 3-dimensional subcomplex of euclidean 3-space E_3 with an even number of boundary 2-simplexes. (2) The latter are partitioned into pairs, and paired elements are related by linear homeomorphisms which induce an equivalence relation (R) among the points of $F_\psi(K_3)$, each equivalence class of an inner point of $F_\psi(K_3)$ being a singleton. The function Φ, defined on pairs of simplexes and on pairs of points, equals 1 for equivalent and 0 for non-equivalent pairs, and $(F_\psi(K_3), \Phi)$ is defined as the quotient space $F_\psi(K_3)/(R)$. In this work, $F_\psi(K_3)$ is always a "σ-figure"; where σ-figures, constructively characterized in the paper, are the most general simply connected pure 3-complexes in E_3. A condition is also imposed implying that K_3 is orientable. "General representations" of K_3 are obtainable from $F_\psi(K_3)$ by (a) identifying fixed elements, where a "fixed element" is a pair (β, γ) of boundary simplexes with $\Phi(\beta, \gamma) = 1$ such that β and γ have a common face on which the linear homeomorphism $\beta \to \gamma$ is the identity, and then (b) using the equivalence relation induced by Φ on the resulting space. A process of "constructing fixed elements" is defined which involves a special category of quotient spaces obtained from K_3, and from $F_\psi(K_3)$, by collapsing certain 2-simplexes. Theorems I and II assert the existence of quotient spaces and ψ-representations with a total of eleven special properties, where K_3 is not necessarily compact in I and is compact in II. In each case, null-homotopic closed curves on K_3 are involved. Theorem III asserts that if K_3 is compact and simply connected, it

admits a general representation (F, Φ) whose boundary reduces to zero. *S. S. Cairns* (Princeton, N.J.)

5197:

James, Ioan; and Thomas, Emery. Which Lie groups are homotopy-abelian? Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 737–740.

The authors call a subgroup H of the topological group G "homotopy abelian in G" if the two maps f and $\bar{f}$ of $H\times H\to G$ defined by $f(x, y)=xy=\bar{f}(y, x)$ are homotopic. Their main result generalizes the theorem of Samelson to the effect that the group of unit quaternions is not homotopy abelian. They show in fact that $\mathrm{U}(n)$ $[\mathrm{Sp}(n)]$ is not homotopy abelian in $\mathrm{U}(2n-1)$ $[\mathrm{Sp}(2n-1)]$ while $\mathrm{SO}(2n)$ $[\mathrm{SO}(2n+1)]$ is not homotopy abelian in $\mathrm{SO}(4n-4)$ $[\mathrm{SO}(4n)]$. Here, of course, $\mathrm{U}(n)$, $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$ denote the unitary, orthogonal and symplectic groups respectively.

Their starting point is similar to Samelson's [Comm. Math. Helv. **28** (1954), 320–327; MR **16**, 389]. According to this paper the commutative map $G\times G\to G: (x, y)\to xyx^{-1}y^{-1}$ defines a product in $\pi_*(G)$—the Samelson product—which under suspension $T: \pi_n(G)\to\pi_{n+1}(B_G)$ goes over into the Whitehead product in the classifying space B_G of G. Now the Samelson product clearly is annihilated by the injection $i: H\to G$ if H is homotopy commutative in G. Hence, if some Whitehead product in B_H is not annihilated by the natural map of B_H into B_G, then H is not homotopy abelian in G. Thus the problem reduces to finding nontrivial Whitehead products in B_G. The authors derive a cohomology criterion, which detects nontrivial squares $[d, d]$ in $\pi_*(B_G)$. This criterion is based on the following double computation principle: Let p be an odd prime and assume that for $d\in\pi_r(B_G)$, r even, $[d, d]$ has an order m, prime to p. Let A_r be the space obtained from $S^r\times S^r$ by identifying the two slices $e\times S^r$ and $S^r\times e$ with S^r. Let $d': S^r\to B_G$ represent md. Then d' extends to a map $f_{d'}: A_r\to B_G$. Because r is even, $H^{2r}(A_r; Z_p)$ is generated by the square of the generator of $H^r(A_r; Z_p)$. One can therefore compute $f_{d'}{}^*$ on decomposable elements of $H^*(B_G; Z_p)$ in dimension $2r$ by knowing d'^*. On the other hand, stable operations are trivial in A_r, as under suspension S^r becomes a retract of A_r. Thus if $x\in H^{2r}(B_G; Z_p)$ is obtained from an element of lesser dimension by a stable operation, then $f_{d'}{}^*x=0$, whence, if the answer using d'^* is not 0, the element $[d, d]$ in question must have an order divisible by p (or be infinite cyclic). This principle of double computation is now applied to the cohomology rings $H^*(B_G; Z_p)$. For the classical groups the necessary information concerning d'^* and the stable operations is available from the work of Borel and Serre.

{The statement of Lemma (2.1) might be clearer if the second sentence of the Lemma started with: "Then no dependent and decomposable element ...".}

R. Bott (Ann Arbor, Mich.)

5198:

Kodama, Yukihiro. On a problem of Alexandroff concerning the dimension of product spaces. I. J. Math. Soc. Japan **10** (1958), 380–404.

The problem was to characterize the finite-dimensional compacta (compact metric spaces) X for which

$$(1)\qquad \dim(X\times Y) = \dim X+\dim Y$$

whenever Y is a compactum. Let a be a sequence of integers $q_1, q_2, \cdots$, for which $1\leqq q_1|q_2|\cdots$ and at least one $q_i>1$; let Z be the group of integers, and let $Z(a)$ denote the inverse limit of the groups Z/q_iZ under the obvious homomorphisms. The main theorem is that for (1) to hold it is necessary (even when Y is restricted to be 2-dimensional) and sufficient that for each a there exists a closed subset A_a of X for which $H_n(X, A_a; Z(a))\neq(0)$ (Čech homology), where $n=\dim X$. In a postscript, the author notes that the problem had already been solved by Boltyanskiĭ [Dokl. Akad. Nauk SSSR **67** (1949), 773–776; MR **11**, 195], in a paper inaccessible to him, and verifies directly that Boltyanskiĭ's criterion (in terms of cohomology) is equivalent to his. Like Boltyanskiĭ's, the present proof depends on using a sequence of "test spaces" Y_i which are generalisations of a space due to Pontryagin. The author also derives some previously known sufficient conditions (e.g., $\dim X=1$, or X a polyhedron) from his criterion, and gives some extensions to locally compact spaces. *A. H. Stone* (Manchester)

5199:

Fuks, D. B.; and Švarc, A. S. Cyclic products of a polyhedron and the imbedding problem. Dokl. Akad. Nauk SSSR **125** (1959), 285–288. (Russian)

Let t be a simplicial homeomorphism of period n of the complex X and let $Y\subseteq X$ consist of those simplexes left fixed by some t^i, $0<i<n$. Let X' be the orbit space of X, $\pi: X\to X'$ the projection, and $Y'=\pi(Y)$. Let s_{2i}, s_{2i-1} be the endomorphisms of $C^*(X; G)$ given by $s_{2i}=1+t+\cdots+t^{n-1}$, $s_{2i-1}=1-t$ and let $s_{2i}{}^*$, $s_{2i-1}{}^*$ be the induced maps of $H^*(X, Y; G)$. The authors define, for each $r>0$, a partial many-valued map $\delta_r: \ker s_i{}^*\to\ker s_{i+r}{}^*$ by declaring that $y\in\delta_r x$ if there is a sequence of cochains $\tau_0, \tau_1, \cdots, \tau_{r-1}\in C^*(X, Y; G)$ such that $s_{i+k}\tau_k$ is the coboundary of τ_{k+1} and τ_0 represents x, $s_{i+r-1}\tau_{r-1}$ represents y. They then assert (theorem 1) that in the spectral sequence of cohomology with compact supports associated with the regular covering $\pi: X-Y\to X'-Y'$ the E_2 term is given by $E_2{}^0=\ker s_1{}^*$, $E_2{}^i=\ker s_{i+1}{}^*/\mathrm{im}\, s_i{}^*$, $i>0$, and the partial map $E_2{}^i\to E_2{}^{i+r}$ (which proceeds via E_r) is just $\phi_{i+r}\delta_r\phi_i{}^{-1}$, where ϕ_i projects $\ker s_{i+1}{}^*$ onto $E_2{}^i$.

The main theorem of the paper utilizes Theorem 1 when $X=K^p$, the p-fold Cartesian power of K; $n=p$; $t: K^p\to K^p$ is the cyclic permutation $x_1, \cdots, x_p\to x_2, \cdots, x_p, x_1$; $G=Z_p$; and Y is the diagonal $i(K)$ in K^p; and in it the authors derive necessary and sufficient conditions for $y\in\delta_r x$ in terms of relations between Steenrod powers of elements appearing in canonical expressions for j^*x, j^*y where $j^*: H^*(X, Y; G)\to H^*(X; G)$. They remark that the results of Nakaoka on the homology of cyclic products [Proc. Japan Acad. **31** (1955), 665–669; MR **19**, 972] are readily deducible from this theorem.

The authors apply the main theorem to the embedding problem. Wu has defined certain cohomology classes $\Phi_p{}^m(K)$, $\Psi_p{}^m(K)$ which are intimately related with the problems of embedding and immersing K in Euclidean space. If $\bar{\Phi}_p{}^m(K)$, $\bar{\Psi}_p{}^m(K)$ are the mod p reductions of these classes it is shown that if K is an n-dimensional manifold and k its fundamental mod 2 cohomology class and if $\bar{\Phi}_2{}^{m-1}(K)\neq0$, $\bar{\Phi}_2{}^m(K)=0$ then (a) $\bar{\Psi}_2{}^{m-2}(K)\neq0$, (b) $\bar{\Psi}_2{}^{m-1}(K)=0$, (c) $\mathrm{Sm}_{m-n-1}(k)\neq0$, (d) $\mathrm{Sm}_j(x)=0$ if $x\in H_s(K; Z_2)$ and $j\geqq m-s$, (e) $\bar{W}_{m-n-1}\neq0$, (f) $\bar{W}_j=0$ for $j\geqq m-n$ (this last condition is misprinted as $j\geqq m-s$ in

the paper). Here Sm_j is the Smith operation, and the $\overline{W}_j$ are the dual Stiefel-Whitney classes. If K is orientable there is a theorem of a similar nature for any odd prime p.
P. J. Hilton (Birmingham)

5200:

Lihtenbaum, L. M. Duality theorem for non-singular graphs. Uspehi Mat. Nauk **13** (1958), no. 5 (83), 185–190. (Russian)

Es sei K ein endlicher, nichtgerichteter, nichtsingulärer (d.i., ohne Zweiecke und Schlingen) Graph mit $\alpha^{(0)}$ Knotenpunkten $a_i^{(0)}$ $(i=1, \cdots, \alpha^{(0)})$ und $\alpha^{(1)}$ Kanten $a_k^{(1)}$ $(k=1, \cdots, \alpha^{(1)})$. Die Zahlen a_{ij} $(i, j=1, \cdots, \alpha^{(0)})$ bzw. b_{kl} $(k, l=1, \cdots, \alpha^{(1)})$, die die Nachbarschaft der Knotenpunkte bzw. der Kanten von K beschreiben, seien folgendermassen definiert: für $i\neq j$ ist $a_{ij}=1$ oder 0, je nachdem zwischen den Knotenpunkten $a_i^{(0)}$ und $a_j^{(0)}$ eine oder keine Kante existiert, a_{ii} ist gleich der Anzahl der mit dem Knotenpunkt $a_i^{(0)}$ inzidenten Kanten von K; für $k\neq l$ ist $b_{kl}=1$ oder 0, je nachdem die Kanten $a_k^{(1)}$ und $a_l^{(1)}$ mit demselben Knotenpunkt inzident sind oder nicht, $b_{kk}=2$ (jede Kante ist mit zwei Knotenpunkten inzident). Es wird bewiesen, dass für jede natürliche Zahl m die Relation

$$\sum_{i,i_1,\cdots,i_{m-1}=1}^{\alpha^{(0)}} a_{ii_1}a_{i_1i_2}\cdots a_{i_{m-1}i} = \sum_{k,k_1,\cdots,k_{m-1}=1}^{\alpha^{(1)}} b_{kk_1}b_{k_1k_2}\cdots b_{k_{m-1}k}$$

gilt. Diese Zahlen $c_m(K)$ werden als Kennzahlen des Graphen K vom Grade m bezeichnet. Auch der geometrische Sinn dieser Kennzahlen (mittels der sog. sich schliessenden Homomorphismen von m-gliedrigen gebrochenen Linien in den Graphen K) wird erörtert.
M. Fiedler (Prague)

5201:

Čulík, Karel. Über Zyklen der zyklischen Graphen. Časopis Pěst. Mat. **83** (1958), 440–450. (Czech. Russian and German summaries)

Ein gerichteter Graph G (eine Menge mit einer binären Relation) heisst zyklisch vom Grade $k\geq 1$, falls aus der Existenz der Kanten $u_1u_2, u_2u_3, u_3u_4, \cdots, u_{k-1}u_k$ in G die Existenz der Kante u_ku_1 in G folgt. Der Grad k eines zyklischen Graphen G wird seine Periode genannt, wenn es in G einen k-kantigen Zyklus gibt, aber keinen l-kantigen Zyklus mit $l<k$. Der Hauptsatz: Ein zusammenhängender gerichteter Graph ist zyklisch mit der Periode $k\geq 3$ dann und nur dann, wenn er ein homomorphes Vorbild eines k-kantigen Zyklus ist, oder ausführlicher, wenn seine Knotenpunkte in k Klassen $U_1, U_2, \cdots, U_k$ so zerlegt werden können, dass zwei Knotenpunkte u, v dann und nur dann durch die Kante uv verbunden sind, falls für einen Index i $(i=1, 2, \cdots, k)$ $u\in U_i$, $v\in U_{i+1}$ gilt (dabei $U_{k+1}=U_1$). [Über den Begriff des Homomorphismus der Graphen s. eine frühere Arbeit des Verf., Časopis Pěst. Mat. **83** (1958), 133–155; MR **20** #4272.]
M. Fiedler (Prague)

5202:

Sedláček, Jiří. Über eine spezielle Klasse wohlgerichteter Graphen. Časopis Pěst. Mat. **84** (1959), 7–15. (Czech. Russian and German summaries)

Ein endlicher gerichteter Graph **D** wird als wohlgerichtet bezeichnet, wenn von jedem Knotenpunkt aus zu jedem anderen Knotenpunkt eine Bahn in **D** führt. Ein Knotenpunkt von **D**, der in jedem Zyklus von **D** enthalten ist, heisst ein Zentrum von **D**. Der Hauptsatz: Ist D ein zusammenhängender, endlicher, nichtgerichteter, artikulationsloser Graph ohne Zweiecke und Schlingen, mit wenigstens drei Knotenpunkten, und c sein beliebiger Knotenpunkt, so existiert eine solche Orientierung sämtlicher Kanten von D, dass ein wohlgerichteter Graph **D** mit dem Zentrum c entsteht.
M. Fiedler (Prague)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 5089, 5159.

5203:

Bruijn, P. J. A theorem on loxodromes. Simon Stevin **32** (1958), 159–161. (Dutch)

Let $1/\tau$ be the torsion and $T=\int ds/\tau$ the total torsion of a curve; then $T=\pi$ [resp. $T=-\pi$] holds for all spherical loxodromic curves which are right [resp. left] rotating, i.e., T is independent of λ and r (the angle between loxodromic and meridian curve, and the radius of the sphere).
S. R. Struik (Cambridge, Mass.)

5204:

Kallenberg, G. W. M. Some generalizations of the theorem on loxodromes by P. J. Bruijn. Simon Stevin **32** (1958), 162–169. (Dutch)

Using known theorems of surface curves, the author improves on the solution of a problem solved by P. J. Bruijn [preceding review]. On a surface $x=x(u, v)$ of three times continuously differentiable parameter functions, is given a line $u=f(v)$, itself 3 times continuously differentiable. A computation of the total torsion T with the help of known formulas [to be found, e.g., in Eisenhart, *Differential geometry*, Princeton Univ. Press, 1947; MR **2**, 154; pp. 193, 227, 247, 249] gives $T=\pi$. On the sphere this holds not only for loxodromic, but for more general curves, of which the author gives 3 examples. The total torsion of a loxodromic curve on an oval surface of revolution leads to values of T, which in the special case of a rotational ellipsoid, $f(r)=(c/a)(a^2-r^2)^{1/2}$, make $T=\pi-\pi(a-c)/a\cdot\sin\lambda$, where λ signifies the angle between meridian and loxodromic curve.
S. R. Struik (Cambridge, Mass.)

5205:

★Fazekas, F. Kombinierte Integralsätze (mit komplexen Veränderlichen) in der zweidimensionalen Feldtheorie. Einige Arbeiten des Lehrstuhles für Mathematik im Lehrjahre 1956/57, pp. 3–7. Wissenschaftliche Veröffentlichungen der Technischen Universität für Bau- und Verkehrswesen in Budapest, Budapest, 1958. 80 pp.

The integral theorems of Stokes and Gauss are applied to the special case of vectors which lie in a plane. If the real and imaginary parts of a regular function $f(z)$ are identified with the two components of these vectors, the following integral theorems hold:

$$\oint_{(G)} \overline{f(z)}\,dz = 2i \iint_{(F)} \overline{f'(z)}\,dF,$$

$$\oint_{(G)} \overline{f(z)}f'(z)\,dz = 2i \iint_{(F)} |f'(z)|^2\,dF,$$

where $\overline{f(z)}$ is the complex conjugate of $f(z)$, G is a rectifiable, continuous curve, and F is the surface bounded by G. *D. E. Spencer* (Storrs, Conn.)

5206:

★Fazekas, F. Bemerkungen über die Ableitungsmatrix und deren Invarianten. Einige Arbeiten des Lehrstuhles für Mathematik im Lehrjahre 1956/57, pp. 29–36. Wissenschaftliche Veröffentlichungen der Technischen Universität für Bau- und Verkehrswesen in Budapest, Budapest, 1958. 80 pp.

The paper shows that the divergence, curl, gradient, and the scalar Laplacian can all be regarded as invariants of certain matrices. Invariants of the derivative tensor $\mathbf{D}=d\mathbf{p}/d\mathbf{r}$ are div $\mathbf{p}$ and **rot** $\mathbf{p}$ while Δu is an invariant of $\mathbf{G}=d^2u/d\mathbf{r}^2$. The vector **grad** u is obtained as the invariant derivative vector $\mathbf{g}=du/d\mathbf{r}$ itself. *D. E. Spencer* (Storrs, Conn.)

5207:

Sparatore, Elio. Su una corrispondenza tra rette dello spazio e linee di una superficie. Atti Accad. Ligure **14** (1958), 107–114. (English summary)

Une surface S est donnée dans l'espace ordinaire. Si r est une droite quelconque de l'espace, il y a sur S une ligne L lieu des points Q de S dont les normales $n(Q)$ rencontrent la droite r. L'équation de L peut s'écrire $(Q-P)\wedge\bar{n}\times\bar{r}=0$, P un point quelconque de r, et $\bar{n}$, $\bar{r}$ des vecteurs unitaires placés respectivement sur n et sur r. L'A. démontre plusieurs propriétés des lignes L; par exemple: Si l'on fixe P dans l'espace, la ligne L varie dans un système linéaire ∞^2; si l'on fixe Q sur S et P sur $n(Q)$, et si l'on veut que la ligne L touche en Q une droite donnée t, alors r doit varier dans un faisceau de centre P dont le plan π passe par $n(Q)$, et qui est projectif au faisceau (d'axe t) des plans osculateurs en Q aux lignes L correspondantes; en faisant varier la droite t tangente en Q à S, les faisceaux décrits par t et par π sont projectifs; les lignes L passant par un point donné Q de S proviennent des droites r du complexe linéaire spécial ayant pour axe $n(Q)$; etc. *E. G. Togliatti* (Genoa)

5208:

Parodi, Maurice. Détermination de courbes planes définies par une inégalité entre les valeurs absolues de fonctions des éléments de contact en un point courant. Bull. Sci. Math. (2) **82** (1958), 14–16.

D'après un théorème classique de M. J. Hadamard, un déterminant $\Delta=|a_{ij}|$ d'ordre n ne saurait être nul si l'on a

$$|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \quad (i = 1, 2, \cdots, n).$$

L'auteur fait de ce théorème une application originale à la théorie des courbes planes. Ayant égard à celles de ces courbes pour lesquelles, $\varphi_i(x, y, y')$ $(i=1, 2, 3)$ étant trois fonctions de l'élément de contact générique, on a (1) $|\varphi_1|\leq|\varphi_2|+|\varphi_3|$, il montre que toute une famille de ces courbes est définie par l'équation différentielle

$$\Delta(x, y, y') = \begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi_2 & \varphi_1 & \varphi_3 \\ \varphi_2 & \varphi_3 & \varphi_1 \end{vmatrix} = 0.$$

Il signale diverses extensions possibles de ce résultat, et traite, comme application, le cas où l'inégalité (1) traduit le fait que, pour les courbes envisagées, la valeur absolue de l'abscisse du point courant est au plus égale à la somme des valeurs absolues de la sous-tangente et de l'ordonnée. Parmi les courbes jouissant de cette propriété figurent, en dehors des solutions triviales constituées par les droites issues de l'origine, la famille d'hyperboles $y^2+2yx-k^2=0$. *P. Vincensini* (Marseille)

5209:

Roman, A. Observations rélatives aux courbes parallèles au sens large. Gaz. Mat. Fiz. Ser. A **10** (**63**) (1958), 641–645. (Romanian. French and Russian summaries)

A differential equation of the Lagrange type $y=x\varphi(p)+\psi(p)$, where $p=dy/dx$, defines a system of curves $x=F_1(p)+tF_2(p)$, $y=G_1(p)+tG_2(p)$, where t is a constant of integration. This system is clinoidal in the sense of A. Myller [Acad. R. P. Romîne. Fil. Iași Stud. Cerc. Şti **5** (1954), 1–15; MR **16**, 620]; conversely, such a system satisfies a Lagrange equation. The envelopes of the lines $p=$const. are studied, which brings in the evolutoids in the sense of Réaumur. *D. J. Struik* (Cambridge, Mass.)

5210:

Miron, R. Une généralisation de la notion de courbure de parallelisme. Gaz. Mat. Fiz. Ser. A **10** (**63**) (1958), 705–708. (Romanian. French and Russian summaries)

When, at the points M of a curve C in ordinary space a unit vector field $\bar{n}$ is defined, and hence a ∞^1 system of planes P perpendicular to $\bar{n}$, then a line l through M in the plane P through M with unit vector $\bar{a}$ is moved parallel along C "in the sense of Levi-Civita", if $d\bar{a}$ is at all points P parallel, in the ordinary sense, to $\bar{n}$, or $d\bar{a}-(\bar{a}\cdot d\bar{n})\bar{n}=0$ [A. Myller, Ann. Sci. Univ. Jassy **13** (1924), 20–34]. The scalar $\pm\bar{a}\cdot d\bar{n}/ds$ has been called, for the case that $\bar{n}$ is the unit normal vector to a surface through C, $\bar{a}=d\bar{x}/\delta s$, the curvature of parallelism of the directions $\bar{a}(\delta s)$ and $\bar{b}(ds)$ [O. Mayer, C. R. Acad. Sci. Paris **178** (1924), 1954–1956]. In the present paper some properties of this scalar are discussed for the case of Myller, with the vector field $\bar{a}$ not necessarily parallel in the sense of Levi-Civita.

D. J. Struik (Cambridge, Mass.)

5211:

Mihăilescu, Tiberiu. Les surfaces R_0. Czechoslovak Math. J. **8** (**83**) (1958), 573–582. (Russian summary)

Les surfaces non réglées projectivement déformables dont le réseau conjugué de déformation projective se réduit à l'une des familles de lignes asymptotiques dépendent de cinq fonctions arbitraires d'un argument [E. Cartan, Ann. Ecole Norm. Sup. (3) **37** (1920), 259–356] et elles sont appelées surfaces R_0. Considérons les repères $A_0A_1A_2A_3$ dont le point A_0 coincide avec le point M de la surface considérée; les droites $[A_0A_1]$, $[A_0A_2]$ sont les tangentes asymptotiques distinctes en A_0 et $[A_0A_3]$, $[A_1A_2]$ sont les directrices de Wilczynski. L'auteur démontre: (a) seules les surfaces R_0 possèdent des réseaux isothermes-conjugués ayant l'axe cuspidal dans un des plans $[A_0A_1A_3]$, $[A_0A_2A_3]$ et ne coincidant pas avec $[A_0A_3]$; (b) si l'axe cuspidal associé à un réseau coincide avec une des droites de Sullivan, le réseau est adhérent a une surface R_0. *L. A. Santaló* (Buenos Aires)

5212:

★**Finikow, S. P. Theorie der Kongruenzen.** Wissenschaftliche Bearbeitung der deutschen Ausgabe: Gerrit Bol. Mathematische Lehrbücher und Monographien, II. Abt., Bd. X. Akademie-Verlag, Berlin, 1959. xvi+491 pp. DM 56.00.

Translation of *Teoriya kongruèncii*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950 [MR **12**, 744].

5213:

Jha, P. On the guiding curves of a rectilinear congruence. Proc. Nat. Inst. Sci. India. Part A **23** (1957), 412–419.

S étant la surface de référence d'une congruence rectiligne (D), l'auteur choisit sur S comme courbes coordonnées u les courbes enveloppées par les projections orthogonales des différents rayons D sur les plans tangents correspondants à S, comme courbes v les trajectoires orthogonales des précédentes, et il étudie les propriétés de (D) en relation avec le paramétrage ainsi choisi. Il envisage divers comportements, l'une par rapport à l'autre, de deux congruences (D) et $(\bar{D})$ réfléchies l'une de l'autre par rapport à S, et examine plus particulièrement le cas où les traces des développables de (D) et $(\bar{D})$ sur S sont les mêmes, orthogonales ou conjuguées. De même sont étudiées les circonstances qui se présentent lorsque les courbes u de S sont géodésiques, ou lorsque les courbes donnant les surfaces de la congruence incidente dont les lignes de striction sont asymptotiques (propriété conservée par reflexion), ou bien lorsque les droites (D) coupent à angle constant la surface S, et, plus particulièrement encore, la surface moyenne de la congruence.

P. Vincensini (Marseille)

5214:

Švec, Alois. Congruences de droites dans les espaces projectifs à dimension paire. Czechoslovak Math. J. **8** (**83**) (1958), 274–284. (Russian summary)

Author's summary: "Dans son Mémoire [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **39** (1953), 481–489; MR **15**, 156] M. Beniamino Segre étudie une classe spéciale de congruences de droites dans l'espace à plusieurs dimensions. Dans le présent travail, je détermine l'élément linéaire projectif des congruences générales de droites, mais en me bornant au cas des espaces à dimension paire. On établit aussi la signification géométrique de la conservation de cet élément et l'on garantit l'existence des congruences pour lesquelles il est donné d'avance. Une théorie analogue peut être développée même pour le cas des congruences des droites plongées dans des espaces projectifs à dimension impaire, comme le fait voir p. ex. mon travail antérieur [Publ. Fac. Sci. Univ. Masaryk **1957**, 87–100; MR **20** #2001]."

P. O. Bell (Culver City, Calif.)

5215:

Urban, Alois. Augmentation du contact des courbes par projection. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **7** (1957), 207–234. (Czech. French summary)

This study of the problem of the augmentation of the order of contact of two curves by projection is a continuation of earlier investigations by Halphen [J. École Polytech. **28** (1880), no. 47, 1–102], Bompiani [Mem. Acad. Sci. Ist. Bologna Cl. Sci. Fis. (8) **3** (1925–26), 35–38; Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (6) **14** (1931), 456–461; Scritti Mat. Off. a L. Berzolari, (1936), 515–552], Čech [*Projektivni diferencialni geometrie*, Nákl. Jedn. Česk. Mat. Fis., Praha, 1926], and the author [Czechoslovak Math. J. **7** (**82**) (1957), 273–293; MR **19**, 675]. In the present paper the author considers the case of two intersecting curves. Known analytic results are obtained by a new method which also leads to a geometrical construction of all points from which the given curves are projected into curves with a contact of the order >0. Moreover, constructions are described of the principal straight lines and the principal points.

H. Schwerdtfeger (Montreal, P.Q.)

5216:

Müller, Hans Robert. Die Formel von Euler und Savary in der affinen Kinematik. Arch. Math. **10** (1959), 71–80.

Com'è noto, nella cinematica classica euleriana la formula di Eulero-Savary fissa una legge di corrispondenza (che è precisamente una trasformazione quadratica birazionale) fra i centri di curvatura di due profili coniugati. Da essa discende, fra l'altro, che, indicando, all'istante t, X il centro di curvatura del profilo solidale con il sistema mobile, ed Y il centro di curvatura del profilo coniugato, il punto Y viene a coincidere con il centro di curvatura, M, della linea luogo dei punti X.

Nel presente lavoro, l'Autore determina la relazione che nella cinematica affine generalizza la formula di Eulero-Savary. La corrispondenza fra i punti X e Y resta ancora una trasformazione quadratica birazionale; poichè non risulta tale la corrispondenza fra i punti X e M, l'Autore conclude che, nella cinematica affine, il punto Y non gode della proprietà di coincidere, in ogni istante, con il centro di curvatura della linea dei punti X.

B. Manfredi (Parma)

5217:

Sorace, Orazio. Intorno ad un problema di W. Blaschke generalizzato. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 465–469.

Let F be a surface in a projective space S_{2r+2} of dimension $2r+2$ $(r\geqq 2)$ satisfying an equation of Laplace. Then the r-dimensional osculating space of F at a general point is an S_{2r}. Following B. Segre, a subspace S_3 of S_{2r+2} is said to be a space of Blaschke for F, if it intersects the r-dimensional osculating spaces S_{2r} of F in the lines of a W congruence with two distinct nondegenerate focal surfaces. In this paper the author obtains necessary and sufficient conditions for a subspace S_3 of S_{2r+2} to be a space of Blaschke of a surface F, and also the parametric equations of a surface F in a space S_{2r+2}, which has every coordinate space S_3 in S_{2r+2} for a space of Blaschke. For the case $r=1$, see B. Segre [Abh. Math. Sem. Univ. Hamburg **20** (1955), 28–40; MR **17**, 1238].

C. C. Hsiung (Madison, Wis.)

5218:

Pyle, H. Randolph. Proportional metrics in n variables. Math. Mag. **32** (1958/59), 261–263.

Linear equations are derived which the transformation of coordinates must satisfy in order that two Riemannian metrics be conformally related. The calculations are purely

formal, and do not go as far as integrability conditions. [Cf. H. Pyle, Duke Math. J. **11** (1944), 369–371; MR **6**, 21.]

L. W. Green (Minneapolis, Minn.)

5219:

Calabi, Eugenio. Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens. Michigan Math. J. **5** (1958), 105–126.

L'A. étudie les solutions localement convexes de l'équation aux dérivées partielles

$$\det\left(\frac{\partial^2 u}{\partial x^i \partial x^j}\right) = 1 \qquad (i, j = 1, 2, \cdots, n)$$

définies dans un domaine D de R^n. Le tenseur $g_{ij} = \partial^2 u/\partial x^i \partial x^j$, associé à une telle solution, définit dans D une métrique riemannienne, et on pose

$$A_{ijk} = -\tfrac{1}{2}\partial^3 u/\partial x^i \partial x^j \partial x^k.$$

Désignant par $\gamma(x)$ la distance géodésique de (x) à la frontière de D, relativement à cette métrique, l'A. établit l'existence d'une constante $c_n < n\sqrt{2}$, donnant lieu à l'inégalité $(A_{ijk}A^{ijk})^{1/2} \leq c_n/\gamma(x)$, ce qui revient à majorer les dérivées troisièmes de u en (x) au moyen d'une majoration uniforme de ses dérivées secondes dans un voisinage de (x). Pour $n \leq 5$, l'A. obtient une majoration analogue, mais faisant intervenir seulement les dérivées secondes de $\dot{u}$ au point (x). La démonstration utilise le principe du maximum généralisé antérieurement établi par l'A. [Duke Math. J. **25** (1958), 45–56; MR **19**, 1056].

J. Lelong (Paris)

5220:

Wakakuwa, Hidekiyo. On Riemannian manifolds with homogeneous holonomy group Sp(n). Tôhoku Math. J. (2) **10** (1958), 274–303.

Some theorems are proved about Riemannian manifolds of dimension $4n$ whose restricted homogeneous holonomy group is a subgroup of Sp(n). This group contains a natural quaternionic subgroup and the corresponding units I, J, K give rise, by parallel translation, to tensors F_1, F_2, F_3 of transformation type, i.e., each F_i assigns at each point of the manifold a linear transformation of the tangent space into itself. It is shown that t, F_1t, F_2t, F_3t are mutually orthogonal and if s is orthogonal to these then so are F_1s, F_2s, F_3s. Conversely, if such F_i exist and are covariant constant then the holonomy group is a subgroup of Sp(n). Invariance properties of the Riemannian curvature, relative to these F_i, are discussed; and the complex version of the above is given.

The second half of the paper discusses consequences of the above for the space of harmonic forms. These F_i induce automorphisms of the space of harmonic p-forms and with corresponding orthogonality properties if p is odd; hence if the number of linearly independent harmonic forms with compact support is finite it is divisible by 4 (if p is odd). If the manifold is compact and the first Betti number $B_1 = 4r$, then the manifold decomposes locally into a product of a flat $4r$-dimensional manifold and a manifold whose holonomy group is a subgroup of Sp($n-r$). Each F_i gives rise to a 2-form and the operation of multiplication by this two form is considered on the space of harmonic forms. The usual discussion of effective forms is then amplified by a consideration of invariance under these operators, obtaining corresponding decomposition theorems.

W. Ambrose (Cambridge, Mass.)

5221:

Kručković, G. I.; and Gu, Čao-Hao. The semireducibility criterion for homogeneous Riemannian spaces. Dokl. Akad. Nauk SSSR **120** (1958), 1183–1186. (Russian)

Ein eigentlicher ($ds^2 > 0$) und homogener riemannscher Raum V_n, stetige Bewegungsgruppe G_r und stationäre Untergruppe H im Punkte M besitzend, wird untersucht. Die Verff. setzen dabei voraus, dass H, betrachtet als eine Drehgruppe im, dem Punkte M entsprechenden, euklidischen Berührungsraum, in ein direktes Produkt von zwei Untergruppen H_0 und H_1 ($H = H_0 \times H_1$), welche offenbar zu zwei senkrecht aufeinander stehenden Hyperebenen E_q und E_{n-q} gehören, zerfällt. Ausgehend dann von einem Satz von H. Wakakuwa [Tôhoku Math. J. (2) **6** (1954), 121–134; MR **16**, 965] leiten die Verff. zwei Sätze ab über die Folgen des erwähnten Zerfalls auf die Schichtung des betrachteten Raumes in komplementäre und aufeinander senkrecht stehende Hyperflächen V_q und V_{n-q}.

T. P. Anđelić (Belgrade)

5222:

Gu, Čao-Hao. On some types of homogeneous Riemannian spaces. Dokl. Akad. Nauk SSSR **122** (1958), 171–174. (Russian)

The cases under consideration are characterized by the fact that the group H of isotropy acting in the tangent space E at a point leaves all directions in a subspace E_0 unchanged and acts in an irreducible way on the orthogonal complement E_1 to E_0. Several canonical forms of ds^2 are derived. See also the paper of G. I. Kručković and the author, reviewed above.

D. J. Struik (Cambridge, Mass.)

5223:

Lelong-Ferrand, Jacqueline. Application des méthodes de Hilbert à l'étude des transformations infinitésimales d'une variété différentiable. Bull. Soc. Math. France **86** (1958), 1–26.

Proofs are given for the results announced in C. R. Acad. Sci. Paris **245** (1957), 1585–1588 [MR **20** #2023]. However, theorem 9a relating the Hilbert space behavior of a vector field X to the point transformation group properties is not correct as stated. {A gap in the proof occurs at the top of page 13.} The results are valid if the transformation group is only required to be defined almost everywhere. The author writes that she is planning to publish a correction.

E. Nelson (Princeton, N.J.)

5224:

Cruceanu, V. Sur les transformations infinitésimales d'un espace de Riemann avec la conservation du volume. Acad. R. P. Romîne. Fil. Iaşi Stud. Cerc. Şti. Mat. **9** (1958), no. 2, 181–188. (Romanian. Russian and French summaries)

According to the summary this note establishes, among others, the following properties of infinitesimal volume-preserving transformations on a Riemannian manifold. (1) If two of the three following conditions are satisfied, the third is also: (a) The vector field ξ determines an infinitesimal volume-preserving transformation; (b) the length of ξ is constant along the trajectories of the field; (c) the variety orthogonal to this field is minimal. (2) If the vectors X_α ($\alpha = 1, 2, \cdots, r$) determine a complete system of r one-parameter groups of volume-preserving

transformations, then they determine an r-parameter group of such transformations. (3) In order that the manifold admit a simply transitive, abelian group of volume-preserving transformations generated by n harmonic vectors, it is necessary and sufficient that there exist local coordinates (x^i) and a function $F(x^i)$ such that the metric tensor $g_{ij}=\partial^2F/\partial x^j\partial x^i$, where the determinant of $\partial^2F/\partial x^j\partial x^i=\text{const.}$ *W. M. Boothby* (St. Louis, Mo.)

5225:

Šapiro, Ya. L. Linear manifolds of geodesic fields of directions in a space with affine connection. Mat. Sb. N.S. **45 (87)** (1958), 511–528. (Russian)

Geodesic direction fields were introduced by the author [Dokl. Akad. Nauk SSSR **32** (1941), 237–239; MR **3**, 191] and also studied by J. Haantjes [Nieuw Arch. Wiskunde (3) **2** (1954), 97–102; MR **16**, 170; see also K. Yano, Proc. Imp. Acad. Tokyo **20** (1944), 284–287; 340–345; MR **7**, 331]. In the present paper the case is taken up in which an affine space with symmetric connection A_{n+m} $(n+m\geqq 3)$ contains m geodesic fields of directions $\overset{a}{A}{}^\sigma$ [see MR **3**, 191], $a=1, 2, \cdots, m$; the A^σ are linearly independent. The conditions are

$$\overset{a}{A}{}^\sigma{}_{,\beta}=\overset{a}{T}\delta^\sigma_\beta+\overset{a}{B}_\beta\overset{a}{A}{}^\sigma,\quad R^\sigma_{\mu(\beta\sigma)}\overset{a}{A}{}^\mu=\overset{a}{H}_{(\beta}\delta^\sigma_{\sigma)}+\overset{a}{F}_{\beta\sigma}\overset{a}{A}{}^\sigma;$$

where the T are scalars, and the B, H, F tensors, $\alpha, \beta, \cdots, \sigma=1, \cdots, m+n$. Such geodesic fields form manifolds A_m, and they can be used to introduce special coordinate systems. The A^σ fields can also be normalised. A number of theorems on these fields are derived and special attention is paid to the Riemannian case.

D. J. Struik (Cambridge, Mass.)

5226:

Reeb, Georges. Foliated structures. Notas Mat. No. 12 (1958), 71 pp. (Portuguese)

This monograph, prepared with the aid of M. M. Peixoto, is an exposition of recent results of Reeb, Ehresmann, Weishu and others on foliated structures. The theory is a generalization of that of dynamical systems, the curve elements of the local structure being replaced by higher dimensional surface elements.

W. Kaplan (Ann Arbor, Mich.)

5227:

Aleksandrov, A. D. Uniqueness theorems for surfaces in the large. VI. Vestnik Leningrad. Univ. **14** (1959), no. 1, 5–13. (Russian. English summary)

[Part V: same Vestnik **13** (1958), no. 19, 5–8; MR **21** #909.] The conditions on the function Φ appearing in the previous articles of this series are generalized and given a more intuitive interpretation. Consider two hypersurfaces S^0, S^1, in E^{n+1} or in an $(n+1)$-dimensional Riemannian space, referred to some parameters $x=(x_1, \cdots, x_n)$. Let these surfaces be given in the form $z^j(x)$ and put $z_i{}^j=\partial z^j/\partial x_i$ $(j=0, 1)$. The principal curvatures of S^j are $k_1{}^j\geqq k_2{}^j\geqq\cdots\geqq k_n{}^j$. Let $\phi(S^j; x)=\phi(f_1k_1{}^j, \cdots, f_nk_n{}^j; x)$, where $f_i=f_i(z^j, z_1{}^j, \cdots, z_n{}^j; x)$, $M>f_i>m>0$, and $\phi(\xi_1, \cdots, \xi_n; x)$ is for each x defined in the ξ_i interval from $\xi_i{}^0=f_i(z^0, z_x{}^0, x)k_i{}^0$ to $\xi_i{}^1=f_i(z^1, z_x{}^0, x)k_i{}^0$, when these are distinct. Moreover, if $\xi_i{}^t=(1-t)\xi_i{}^0+t\xi_i{}^1$, it is assumed that $C>\int_0^1\{\partial\phi(\xi_1{}^t, \cdots, \xi_n{}^t; x)/\partial\xi_i{}^t\}dt>c>0$, where C and c are independent of x and i. Then $\Delta\phi=\phi(S^1; x)-\phi(S^0; x)$ is an elliptic differential expression in terms of $\Delta z=z^1(x)-z^0(x)$. The implications of this ellipticity are analogous to those given in the preceding papers.

With $\Delta\xi_i=\xi_i{}^1-\xi_i{}^0$ the intuitive interpretation of the ellipticity is this: if $\Delta^+(x)=\max(\Delta\xi_1, \cdots, \Delta\xi_n, 0)$, $\Delta^-(x)=-\min(\Delta\xi_1, \cdots, \Delta\xi_n, 0)$, then $\Delta\phi\geqq 0$ $[\Delta\phi\leqq 0]$ is equivalent to the existence of a positive constant A with $A\Delta^+(x)\geqq\Delta^-(x)$ $[\Delta^+(x)\leqq A\Delta^-(x)]$. For example, if $f_i\equiv 1$, hence $\Delta\xi_i=k_i{}^1-k_i{}^0$, then $\Delta\phi$ is for convex hypersurfaces S^0, S^1 in E^{n+1} equivalent to the statement that the Dupin indicatrix at x of S^1 fits inside or on that of S^0.

H. Busemann (Los Angeles, Calif.)

5228:

Rembs, E.; e Grotemeyer, K. P. Deformabilità delle superficie in grande e in piccolo. Rend. Mat. e Appl. (5) **17** (1958), 262–304.

This is a detailed survey—with some new or improved results which cannot be discussed here—of the book *Flächenverbiegung im Grossen* [Akademie-Verlag, Berlin, 1957; MR **19**, 59], the first part of which consists of a translation from the Russian article by N. V. Efimov, Uspehi Mat. Nauk (N.S.) **3** (1948), no. 2 (24), 47–158 [MR **10**, 324] and the second, of commentaries by Rembs and Grotemeyer reporting on newer results partly due to them.

H. Busemann (Los Angeles, Calif.)

PROBABILITY

See also 4888, 4945, 5148, 5274, 5399, 5519.

5229:

Fisher, Ronald A. Mathematical probability in the natural sciences. Technometrics **1** (1959), 21–29.

Presidential address given at Symposium III of the XVIIIth Congrès International des Sciences Pharmaceutiques. Identical with the paper in Metrika **2** (1959), 1–10 [MR **21** #935].

5230:

Narayana, T. V. A partial order and its applications to probability theory. Sankhyā **21** (1959), 91–98.

The partial order in question is that induced by a relation of dominance among r-part compositions, that is, ordered sets of r integers with given sum. A composition $(t_1, t_2, \cdots, t_r)$ dominates $(t_1', t_2', \cdots, t_r')$ if each of the partial sums $t_1, t_1+t_2, \cdots$ is at least as large as its primed correspondent. Each pair of compositions is in 1-1 relation to a set of election returns for two candidates and r election districts in which one candidate is never behind (and neither wins!). Hence the number of dominances among r-part compositions of n is equal to the number of lattice paths from (0, 0) to (n, n) which have $2r$ segments and do not cross the line from (0, 0) to (n, n); this number is shown to be $n^{-1}\binom{n}{r}\binom{n}{r-1}$. More generally the corresponding number of lattice paths from (0, 0) to (n, m) is found to be

$$\binom{n-1}{r-1}\binom{m-1}{r-1}-\binom{n-1}{r-2}\binom{m-1}{r}.$$

The similar number in k dimensions with terminal point $(n_1, n_2, \cdots, n_k)$ is said to be given by the determinant $|A_{ij}|$, $i, j=1, 2, \cdots, k$, and $A_{ij}=\binom{n_j-1}{r+j-i-1}$. These results are related to the election problem stated above but with more general conditions and also to the following coin tossing problem. Two coins with probabilities of head p_1 and p_2, respectively, are tossed according to the rules; the first is with coin 1, each succeeding toss is made with coin 1 if its predecessor was a tail and with coin 2 otherwise, and the game is completed when for the first time the number of heads exceeds the number of tails by 2.

J. Riordan (New York, N.Y.)

5231:

Rihter, V. [Richter, Wolfgang]. Precision of an inequality of S. N. Bernstein for large deviations. Vestnik Leningrad. Univ. **14** (1959), no. 1, 24–29. (Russian. English summary)

In the case of large deviations the author proves two inequalities for the distribution functions of the normed sums Z_n of independent random variables X_j and of the normed sums Z^n of independently identically distributed random vectors $X^{(j)}$. An inequality of S. N. Bernstein [*Teoriya veroyatnosteĭ*, Moscow, 1946, p. 162] initiated this investigation. In the multidimensional case only the probability that the normed sum $Z^{(n)}$ takes a value in a half-space is estimated, since the estimation of such probabilities in the case of large deviations in more general sets remains unsolved.

H. P. Edmundson (Pacific Palisades, Calif.)

5232:

Mamaĭ, L. V. Some theorems of the theory of positive-definite functions. Dokl. Akad. Nauk SSSR **126** (1959), 271–273. (Russian)

Let $f_1, f_2, \cdots$ be characteristic functions for which (1) $\prod_1^\infty f_j^{\alpha_j}=\phi(t)$ in some (real) neighborhood of the origin, where ϕ is a function of a complex variable, regular in the strip $|\mathrm{Im}\, t|<M_0$, not vanishing there, and $\inf_j \alpha_j>0$. Then each f_j can be extended to be regular in the same strip, and (1) will be valid there. The same conclusion holds if (1) is only supposed true on a sequence of points converging to 0, if in addition it is supposed that there is some neighborhood of 0 in which no f_j vanishes. This theorem generalizes to the infinite product in (1) results obtained for finite products by Zinger and Linnik [Vestnik Leningrad. Univ. **10** (1955), no. 11, 51–56; MR **17**, 753].

J. L. Doob (Urbana, Ill.)

5233:

Fisz, M. On necessary and sufficient conditions for the validity of the strong law of large numbers expressed in terms of moments. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 221–225. (Russian summary, unbound insert)

According to the views lately developed by Prochorov, concerning the expression in terms of moments of the necessary and sufficient condition for the validity of the strong law of large numbers (SLLN), the author establishes two negative theorems: I. For sequences of independent, symmetric random variables X_i (satisfying the relations $|X_i|<i$ $(i=1, 2, \cdots)$), no necessary and sufficient conditions expressed in terms of the variances σ_i^2 of X_i are existing for the SLLN. II. Let r be an arbitrary given natural number. There do not exist for arbitrary sequences of independent random variables necessary and sufficient conditions for the validity of the SLLN expressed in terms of the moments $m_{i1}, m_{i2}, \cdots, m_{ir}$.

O. Onicescu (Bucharest)

5234a:

Petrov, V. V. Asymptotic expansions for distributions of sums of independent random variables. Teor. Veroyatnost. i Primenen. **4** (1959), 220–224. (Russian. English summary)

5234b:

Petrov, V. V. On a class of limit theorems for independent random variables. Teor. Veroyatnost. i Primenen. **4** (1959), 225–228. (Russian. English summary)

In the first paper, Liapunov-Cramér type of expansions is given for the distribution and density functions of sums of non-identically distributed independent random variables. In the second paper, the rate of convergence to the normal distribution in L_2-norm is given. The condition

$$\int_{|t|>\varepsilon} \prod_{j=1}^n |f_j(t)|^\delta dt = o(n^{-\alpha})$$

for $\delta=1$ or 2, every $\varepsilon>0$ and a certain appropriate α is imposed. No novelty of attack is in sight.

K. L. Chung (Syracuse, N.Y.)

5235:

Ghurye, S. G. A remark on stable laws. Skand. Aktuarietidskr. **1958**, 68–70 (1959).

Let $n\geqq 2$ and let $f_1(t), \cdots, f_n(t), g(t)$ be nondegenerate characteristic functions. Let $\alpha_1, \cdots, \alpha_n$ be positive numbers and suppose there exists a function λ of n variables such that for all positive $b_1, \cdots, b_n$ and all t we have $\prod_{i=1}^n f_i^{\alpha_i}(b_it)=g[\lambda(b)t]$ where $\lambda(b)=\lambda(b_1, \cdots, b_n)$. Then $g(t)$ is stable and there exist positive numbers $c_1, \cdots, c_n$ such that $f_i(t)=g(c_it)$ for $i=1, \cdots, n$ and all t.

J. R. Blum (Albuquerque, N.M.)

5236:

Dugué, D. Sur la convergence presque certaine des séries aléatoires. J. Math. Pures Appl. (9) **38** (1959), 267–273.

It is shown that when $\{X_i\}$ are independent random variables with $\sum E|X_i|^\alpha<\infty$, then if $0<\alpha\leqq 2$ $(1<\alpha\leqq 2$ and $\sum |EX_i|<\infty)$, $\sum X_i$ converges when centered with probability one (converges with probability one). An α-analogue of Kolmogoroff's inequality is also given.

H. Teicher (Lafayette, Ind.)

5237:

Geffroy, Jean. Quelques extensions du théorème de M. Paul Lévy sur la convergence presque sûre des séries aléatoires à termes indépendants. C. R. Acad. Sci. Paris **249** (1959), 1180–1182.

"M. Paul Lévy a démontré que, pour une série aléatoire numérique à termes indépendants, la convergence en probabilité entraîne la convergence presque sûre. Nous montrons ici que cette propriété est encore vraie dans un espace de Banach, ainsi que pour certaines séries à termes liés."

Résumé de l'auteur

5238:

Lévy, Paul. Sur quelques classes de fonctions aléatoires. J. Math. Pures Appl. (9) **38** (1959), 1–23.

The present paper is a continuation of the author's extensive study of Laplacian processes and of the more general processes with linear correlation. (E.g., see his paper, Illinois J. Math. **1** (1957), 217–258 [MR **20** #7339], and the many references given therein.) With respect to Laplacian processes, the author studies processes $\{\Phi(t); t \geqq 0\}$ representable by

$$(*) \qquad \Phi(t) = \int_0^t F(t, u)dX(u),$$

where $\{X(u); u \geqq 0\}$ is an additive Laplacian process with zero mean and the kernel F is an arbitrary function for which the integral and its covariance exists. Such representations are by no means unique, as is emphasized in his earlier papers. In the present paper the author defines (*) to be a canonical representation if, for $t < t'$,

$$E\{\Phi(t')|\Phi(u);\ 0 \leq u \leq t\} = \int_0^t F(t', u)dX(u).$$

Such a canonical representation is unique in that the differential $F(t, u)dX(u)$ is completely determined (le théorème d'unicité). It is proven that there always exists an equivalent representation of (*) in which $\{X(u); u \geqq 0\}$ is Brownian motion. The author then studies the problem of obtaining sufficient conditions for the class of processes of the form (*) to be closed under addition of (independent) processes in the class.

In the last three sections the author returns to his study of the more general processes with linear correlation. The problem is as follows. Consider the class C' of processes having representations of the form $\Phi(t) = \phi(t) + \int_0^t F(t, u)dZ(u)$ where ϕ is a sure function and $\{Z(u); u \geqq 0\}$ is an additive process (not necessarily Laplacian). Let $Z(u) = f(u) + X(u) + Y(u) + S(u)$ be the decomposition of the Z-process into a sure function, a Laplacian process, a continuous process without a Laplacian component and a sum of fixed discontinuities, respectively. Let C be the class of processes representable as

$$(**) \qquad \Phi(t) = \phi(t) + \int_0^t [F(t, u)dX(u) + G(t, u)dY(u) + H(t, u)dS(u)].$$

Let a subscript Z denote the corresponding class of processes formed from the same Z-process. The author's main theorem is that $C = C'$. It is also shown that C_Z and C_Z' need not be equal.

In the paper referred to at the beginning of this review, the author states that the class K_t^*, of all processes on $(0, \infty)$ which at each t are linear functionals of the values of some additive process over the interval $(0, t)$, is equal to the closure of C, $\bar{C}$, and that this would be proven elsewhere. In the present paper the author states that this equality may be easily proven, and poses the more difficult problem of whether or not $C = \bar{C}$. This problem is not resolved, but the conjecture is strengthened by the result that an a.s. convergent sequence of independent members of C belongs to $\bar{C}$. On the other hand, the author proves that it is possible to find a kernel and an additive process such that the class of processes of the form

$$\Phi(t) = \phi(t) + \int_0^t F(t, u)d[Z(u) - g(u)],$$

for fixed F and Z, is not closed, "théorème qui peut rendre l'hypothèse contraire assez plausible". The author will continue to study this problem, as well as that of obtaining a canonical representation and unicity theorem for processes of the form (**).

R. Pyke (New York, N.Y.)

5239:

Feller, William. Notes to my paper "On boundaries and lateral conditions for the Kolmogorov differential equations." Ann. of Math. (2) **68** (1958), 735–736.

Corrections and clarifications to the cited paper [same Ann. **65** (1957), 527–570; MR **19**, 892].

G. E. H. Reuter (Durham)

5240:

Doob, J. L. Brownian motion on a Green space. Teor. Veroyatnost. i Primenen. **2** (1957), 3–33. (Russian summary)

A space R, locally isometric to an open N-dimensional sphere is called a Green space of dimension N. If R is a Green space of dimension N then $R(\pm)$ is the direct product of R with the real line. A heat space is an open subset of $R(\pm)$. Brownian motion processes are defined on Green spaces and heat motion processes are defined on heat spaces. The Brownian motion is defined first and the heat process is defined by means of these processes just as in the Euclidean case. To define a Brownian motion process, the author first defines a Brownian motion transition function and proves that for any Green space there is a unique such function. It is proved that this transition function can be represented by a density function satisfying the Chapman-Kolmogorov equation and is symmetric in the space variables. This was proved by Hunt [Trans. Amer. Math. Soc. **81** (1956), 294–319; MR **18**, 77] when R is a domain of Euclidean N-space.

A Brownian motion process on a Green space has a lifetime τ (possibly ∞) uniquely determined by the value of the variance parameter and the initial distribution. In the special case where R is an open connected set of Euclidean N-space, considering R as a Green space, the Brownian motion obtained is the usual Brownian motion stopped if it hits the boundary of R. The lifetime is the time required to reach the boundary—or ∞ if the boundary is not reached. Subharmonic functions on Green spaces and subparabolic functions on heat spaces are defined by requiring that they have the usual property locally.

The connections between Brownian motion and subharmonic functions developed by the author [ibid. **77** (1954), 86–121; MR **16**, 269], and between heat processes and subparabolic functions [ibid. **80** (1955), 216–280; MR **18**, 76], are exploited and extended to corresponding results for Green spaces and heat spaces. For example, the following theorem is proved.

Let u be a subparabolic [subharmonic] function on a heat [Green] space R, and suppose that $u \leqq v$ on R, where v is parabolic [harmonic] and positive on R. Let $\{z(t), t \geqq 0\}$ be a heat [Brownian] motion process on R, with lifetime τ. Then $\lim_{t \to \tau} u[z(t, w)]$ exists and is finite with probability 1.

It is proved that almost every path of a heat motion process on a heat space approaches ∞ (stays in any compact subset only a finite length of time) as the parameter value approaches the path lifetime. If $p(t, \xi, \eta)$ is the density for a Brownian motion on a Green space then,

for fixed η, p defines a positive superparabolic function on $R(\pm)$ and thus has a limit on almost all heat paths to ∞ of every heat process on $R(\pm)$. The limit is proved to be 0.

A Green space is said to have null boundary if the only subharmonic functions on it which are bounded from above are the constant functions. If R does not have a null boundary, it is said to have a positive boundary. The behavior of a Brownian motion process is shown to be quite different in the two cases. For example, if R is a Green space with a null boundary, then the lifetime of every Brownian motion process on R is $+\infty$ with probability 1 and almost every path of such a process is everywhere dense. If R has a positive boundary and has dimension $\geqq 2$, then almost every path of a Brownian motion on R is nowhere dense and converges to ∞ as its parameter value approaches the path lifetime. An example of the first case is the Riemann surface of the logarithm function, spread over the finite plane less one point. An example of the second case is the Riemann surface of the elliptic modular function, spread over the finite plane less two points.

The Dirichlet problem on Green spaces is discussed from a probabilistic point of view along the lines of the author's previous work for the Euclidean case in the papers cited above, and for more general processes in #5242 below and reference therein.

The paper ends with a discussion of Green's functions and a brief discussion of potential theory for Green spaces.

J. L. Snell (Hanover, N.H.)

5241:

Doob, J. L. La théorie des probabilités et le premier problème des fonctions frontières. Publ. Inst. Statist. Univ. Paris **6** (1957), 289–290.

Author's introduction: "Ce qui suit est un précis, avec quelques compléments, des résultats probabilistiques d'un article dédié à M. Paul Lévy qui paraîtra dans l'Illinois Journal of Mathematics." [Review below.]

H. Cramér (Stockholm)

5242:

Doob, J. L. Probability theory and the first boundary value problem. Illinois J. Math. **2** (1958), 19–36.

L'A. continue son étude [Proc. 3rd Berkeley Symp. Math. Statist. Probab. 1954–1955, vol. 2, p. 49–80, Univ. of California Press, Berkeley, 1956; MR **18**, 941] à l'analyse de laquelle nous renvoyons. Il reprend les mêmes problèmes dans l'espace R en les relativisant, c'est-à-dire en considérant, au lieu des fonctions régulières, leurs quotients par une fonction régulière $h>0$ fixe, ou plutôt lorsque h est seulement $\geqq 0$, des fonctions u dites h-régulières, telles que uh soit régulière; d'où, sur la frontière des domaines réguliers, une mesure μ_ξ^h dont l'intégrale $\int u d\mu_\xi^h$ vaut $u(\xi)$, si ξ est un point de l'ensemble H où $h>0$. Cette mesure s'étend à des domaines D relativement compacts, fortement PWB résolutifs pour lesquels l'axiome $M(D, D')$ est satisfait (axiome remplaçant un axiome de maximum), donc aux domaines fondamentaux R_n dont la réunion est R. On introduit aussi les fonctions h-superrégulières. Les extensions seraient banales si $h\neq 0$ partout, mais il est toujours intéressant de comparer les problèmes relatifs à des h différents. On introduit les h-trajectoires analogues à celles du cas $h=1$, mais avec la mesure μ^h. On designe par D^h la classe des fonctions h-régulières u telles que pour la suite R_n, l'intégrale de $|u|$ pour la mesure correspondante $(\mu_\xi^h)_n$ (ξ fixé dans H) sur l'ensemble où $u>a$, tend vers 0 quand $a\to\infty$, uniformément en n.

Une étude nouvelle concerne les fonctions h-minimales u, c'est-à-dire $\geqq 0$, h-régulières et telles que toute autre fonction h-régulière u_1 satisfaisant à $0\leqq u_1\leqq u$ sur H, est proportionnelle à u sur H. On montre en particulier que pour une telle fonction u, ou bien (a) $u\notin D^h$ et possède la limite 0 sur presque toute h-trajectoire issue de chaque point de H, ou bien (b) il existe $a>0$, majorant u sur H, tel que u tend vers 0 ou a sur presque toute h-trajectoire issue de tout $\xi\in H$, et que $u(\xi)/a$ est la probabilité que la limite soit a.

D'autre part, l'A. introduit une frontière R' telle que $R\cup R'$ est compact séparable. Si A est un ensemble k_σ de R', on considère la probabilité $u_A{}^h(\xi)$ pour qu'une h-trajectoire issue de $\xi\in H$ admette un point d'accumulation sur A. Cela amène l'auteur à introduire la probabilité de rencontrer un ouvert de R, et même pour une h-trajectoire issue de ξ et continue à droite, la probabilité $u_G{}^h(\xi)$, si elle existe, pour qu'elle rencontre G ouvert $\subset R$. Dans les applications en vue, elle définirait alors une fonction h-superrégulière de ξ, $\leqq 1$ sur H, égale à 1 dans $G\cap H$, h-régulière hors $\bar{G}$ dans H, de limite 1 sur presque toute h-trajectoire rencontrant G une infinité de fois, de limite 0 sur presque toute autre trajectoire; de plus elle minore sur H toute $u\geqq 0$ superrégulière, $\geqq 1$ sur G. Cela conduit l'auteur à l'hypothèse J^h qu'il existe une fonction $u_G{}^h$ jouissant des propriétés précédentes au moins pour une suite d'ouverts $G_n\subset R$, qui sont les intersections avec R d'ouverts décroissant dans $R\cup R'$ vers une limite A, ensemble fermé de R'. Il en tire une série de conséquences, en particulier pour interpréter le problème de Dirichlet selon la méthode (PWB). Ainsi, étendant le cas $h=1$, si R' est fortement (PWB)^h résolutive, presque toute h-trajectoire issue d'un point de H converge; de plus, ce qui est nouveau, la réciproque est vraie moyennant J^h.

M. Brelot (Paris)

5243:

Sjöberg, Boris. Über Brownsche Bewegung mit Absorptionsschranken. Acta Acad. Abo. **21**, no. 14, 12 pp. (1959).

Let x_t $(t=0, 1, \cdots)$ be the usual one-step Bernoulli random walk over the integers with probabilities p and $q=1-p$ for respective unit steps to the right and left. In a previous paper [same Acta **21** (1958), no. 13; MR **21** #373] the author calculated the n-step transition probabilities in the presence of two absorbing barriers, and the density of particles generated by a constant source which independently undergo the above random work. In the present paper the limiting form of these solutions is obtained under the usual norming on time, space, p and q. These limits are the well known solutions to the corresponding constant-coefficient diffusion equation.

D. A. Darling (Ann Arbor, Mich.)

5244:

Datzeff, A. B. Über einen Fall der Molekülbewegung. C. R. Acad. Bulgare Sci. **12** (1959), 105–108. (Russian summary)

It is supposed that molecules are moving in a rectangle and if they reach the wall they reflect elastically. At time $t=0$ all the molecules are found with constant linear density on the circumference of a circle and every molecule

has the same velocity, which is radial in direction. The author proves that if $t\to\infty$, then the distribution of the molecules in the rectangle approaches to the uniform distribution. *L. Takács* (New York, N.Y.)

5245:

Harrison, Gerald. Stationary single-server queuing processes with a finite number of sources. Operations Res. 7 (1959), 458–467.

"Some basic relations are obtained that apply to finite stationary queuing processes. ... When the source idle time follows a negative exponential distribution, the service time distribution being arbitrary, the length of the waiting line [for a single server] at instants of completion of service is a Markov chain. Its distribution as well as the waiting-time distribution is obtained. The results are specialized to the cases of negative exponential and constant service times." (From the author's summary)

D. G. Kendall (Oxford)

5246:

Ventcel', A. D. On lateral conditions for multidimensional diffusion processes. Teor. Veroyatnost. i Primenen. 4 (1959), 172–185. (Russian. English summary)

Let $\mathfrak{A}$ be the elliptic operator

$$\mathfrak{A}F(x) = \sum_i \sum_j A_{ij}(x) \frac{\partial^2 F(x)}{\partial\xi_i \partial\xi_j} + \sum_i B_i(x) \frac{\partial F(x)}{\partial \xi_i}$$

in a closed bounded subset K in n-dimensional Euclidean space. In this paper the author is concerned with finding the most general lateral conditions (i.e., conditions under which the range of the resolvent operator coincides with a prescribed set) such that $\mathfrak{A}$ is the infinitesimal generator of a Markovian semi-group; that is, such that $\mathfrak{A}$ generates a semi-group of operators $\{T_t, t \geqq 0\}$, with

$$T_t f(x) = \int_K f(y) P(t, x, dy),$$

where $P(t, x, M)$ is the transition probability function of the process. When K is a circle, or a sphere, this problem is solved for processes which are rotation invariant. The general case is also considered; and in this case lateral conditions are found which are satisfied by all smooth functions in the domain of $\mathfrak{A}$.

A. T. Bharucha-Reid (Eugene, Ore.)

STATISTICS

See also 5499, 5505, 5515, 5518, 5519, 5520, 5525.

5247:

Kudô, A. On the distribution of the maximum value of an equally correlated sample from a normal population. Sankhyā 20 (1958), 309–316.

Let $(x_1, x_2, \cdots, x_N)$ have a multivariate normal distribution with equal means, equal variances and equal correlation coefficients ρ. Without recognizing that (when $\rho \geqq 0$) we can write $x_i = y_i + z$, where the $N+1$ variables $y_1, y_2, \cdots, y_n, z$ are normal and independent, the author expresses the distribution function of max x_i in terms of a series of functions, some of which he tabulates. As an example of this distribution, the author considers the statistic $(x_M - \bar{x})/\sigma$, where $\bar{x}$ is the average of $N_1 + N_2$ independent variables having a common normal distribution with variance σ^2, and x_M is the maximum of the first N_1 of these variables. *F. J. Anscombe* (Chicago, Ill.)

5248:

★Fisher, R. A. Recent progress in experimental design. L'application du calcul des probabilités. Colloque tenu à Genève, 12–15 juillet 1939, pp. 19–31. Collection Scientifique. Institut International de Coopération Intellectuelle, Pâris, 1945. 276 pp. 10 francs suisses.

Expository and polemical paper delivered 20 years ago covering the familiar experimental designs. From the historical point of view it is interesting that in advocating randomization the author stresses its importance for estimation of the sampling error but not directly as means of ascertaining that any observed effects be ascribable to the treatments rather than to some other causes.

J. Neyman (Berkeley, Calif.)

5249:

★Wilks, S. S. De l'application de la théorie des probabilités à certains problèmes d'échantillonnage en statistique mathématique. L'application du calcul des probabilités. Colloque tenu à Genève, 12–15 juillet 1939, pp. 33–80. Collection Scientifique. Institut International de Coopération Intellectuelle, Paris, 1945. 276 pp. 10 francs suisses.

Expository paper delivered at an international conference organised on the eve of the World War II. Likelihood ratio tests, normal and multinomial distributions.

J. Neyman (Berkeley, Calif.)

5250:

★Neyman, J. Basic ideas and some recent results of the theory of testing statistical hypotheses. L'application du calcul des probabilités. Colloque tenu à Genève, 12–15 juillet 1939, pp. 81–127. Collection scientifique. Institut International de Coopération Intellectuelle, Paris, 1945. 276 pp. 10 francs suisses.

As indicated in the heading, this paper was presented at a conference on the applications of probability theory held under the auspices of the International Institute of Intellectual Cooperation (an agency of the League of Nations) in Geneva in July, 1939. The nine papers presented at this Conference were not published until 1945.

The present paper is a résumé as of 1939 of the now classical theory of testing statistical hypotheses, as formulated by J. Neyman and E. S. Pearson. The paper contains an introductory discussion of the difficulties involved in the use of Bayes' formula in the testing of hypotheses as proposed by Jeffreys, and how the testing of hypotheses can be approached on the basis of classical probability theory.

The paper gives an excellent summary of the basic ideas (together with some results) of the Neyman-Pearson theory of hypothesis-testing as it existed in 1939 concerning such notions as errors of first and second kind, critical region of a test, power function of a test, unbiased test, uniformly most powerful test, and acceptance regions similar to the sample space. These concepts depend almost entirely on having situations in which the probability distribution of the underlying random variables depends on one or more (real) parameters. The

author points to difficulties involved in applying the general Neyman-Pearson theory to situations in which much less is known about the probability distribution, particularly those in which nothing more can be assumed about the distribution than possibly its symmetry with respect to the random variables, or symmetry and independence of random variables. He points to Fisher's principle of randomization as a way out for hypothesis-testing in these problems—a direction which has been vigorously developed by many authors during the last fifteen to twenty years into a field now known as non-parametric hypothesis-testing.

S. S. Wilks (Princeton, N.J.)

5251:

Siotani, Minoru. The extreme value of the generalized distances of the individual points in the multivariate normal sample. Ann. Inst. Statist. Math. Tokyo. **10** (1959), 183–208.

Let $y_\alpha' = (y_{1\alpha}, y_{2\alpha}, \cdots, y_{p\alpha})$ $(\alpha = 1, 2, \cdots, N)$ be N p-variate vectors with a zero mean vector and covariance matrix $\gamma\Lambda$ $(\gamma > 0)$, and let the covariance matrix of y_α and y_β $(\alpha \neq \beta)$ be $\delta\Lambda$, where $|\delta| < \gamma$ and Λ is a symmetric, positive definite matrix. The generalized distance of the αth variate from the origin is $\chi_\alpha^2 = \gamma^{-1} y_\alpha' \Lambda^{-1} y_\alpha$. The studentized form is $T_\alpha^2 = \gamma^{-1} y_\alpha' L^{-1} y_\alpha$, where L is an unbiased estimator of Λ independent of y_α and based on ν degrees of freedom. This paper is concerned with the upper percentage points of the maximum values (with respect to α) of $\gamma\chi_\alpha^2$ and γT_α^2, designated by $\hat{\chi}^2_{\max}$ and $\hat{T}^2_{\max}$.

Certain special cases are considered involving a random sample of size n: (i) Maximum deviate from the population mean; (ii) maximum deviate from the sample mean; (iii) maximum deviate from a control variate; (iv) positive square root of the maximum of the squares of generalized distances between any two observed points. In (iv), $N = n(n-1)/2$; $N = n$ for the first three cases.

Let

$$\Pr(\hat{\chi}_1^2 > A_1^2) = P_1; \Pr(\hat{\chi}_1^2 > A_1^2, \chi_2^2 > A_1^2) = P_{12}; \cdots.$$

Then

$$\Pr(\hat{\chi}^2_{\max} > A_1^2) = P = NP_1 - \tfrac{1}{2}N(N-1)P_{12} + \cdots.$$

If $NP_1 = \alpha$, $\alpha - \beta < P < \alpha$. $\beta = \frac{1}{2}N(N-1)P_{12}$ is evaluated from a two-dimensional chi-square distribution. Upper percentage points, A_2^2, are presented for case (ii), for which $N = n$, $\gamma = (n-1)/n$ and $\delta = -1/n$. These are computed so that $\Pr(\hat{\chi}_1^2 > A_2^2) = \alpha + \beta$, for $n = 3(1)10(2)20$, 25, 30; $p = 2, 3, 4$; $\alpha = 0.05, 0.025, 0.01$.

Asymptotic results (up to order ν^{-2}) are obtained for $\hat{T}^2_{\max}$. Values of B_2^2 are computed for $n = 3(1)12$, 14; $\nu = 20(2)40(5)60$, 100, 150, 200; presumably (but not explicitly stated) for $p = 2$.

R. L. Anderson (Raleigh, N.C.)

5252:

Aggarwal, Om P.; and Guttman, Irwin. Tables of the cumulative distribution functions of samples from symmetrically truncated normal distributions. Ann. Inst. Statist. Math. Tokyo **11** (1959), 55–68.

Let $\bar{Z}$ be the arithmetic mean of a random sample of size n from a $N(0, 1)$ distribution truncated to the interval $(-a, a)$, and $\Phi_n(t) = \text{Prob}\,[0 \leqq \bar{Z} \leqq t]$. For $n = 1, 2, 3, 4$ the values of $\Phi_n(t)$ to five significant figures are tabulated, corresponding to $a = 1(0.5)3$ with $t = 0.01(0.01)a$.

F. C. Andrews (Eugene, Ore.)

5253:

Burkholder, D. L. On the existence of a best approximation of one distribution function by another of a given type. Ann. Math. Statist. **30** (1959), 738–742.

"Suppose that F and G are distribution functions and that an ordered real number pair (a, b), with $a > 0$, is desired such that $F(ax+b)$ is close to $G(x)$ for all real x. Is there a best pair? For instance, is there a pair (a_0, b_0) satisfying

$$(1) \quad \sup_{-\infty<x<\infty} |F(a_0x+b_0) - G(x)| = \inf_{\substack{0<a<\infty \\ -\infty<b<\infty}} \sup_{-\infty<x<\infty} |F(ax+b) - G(x)|?$$

In this note we give an example in which a pair (a_0, b_0) satisfying (1) does not exist. We then prove two theorems each giving a simple sufficient condition for the existence of such a pair. One or the other condition is almost always satisfied in practice. For example, the first requires, merely, that both of the sets $\{x \mid 1/3 \leqq F(x) \leqq 2/3\}$ and $\{x \mid 1/3 \leqq G(x) \leqq 2/3\}$ be nondegenerate. Next, we show that in any case if the set of minimizing pairs is nonempty then it is convex. This fact is used to obtain a fairly precise description of the set of minimizing pairs for the case F is increasing and continuous. In this case, simple conditions on G imply the uniqueness of a minimizing pair. Applications, especially to an estimation problem involving an unknown scale and location parameter, are then discussed." (From the author's summary)

J. Wolfowitz (Ithaca, N.Y.)

5254:

Rao, B. R. On an analogue of Cramér-Rao's inequality. Skand. Aktuarietidskr. **1958**, 57–64 (1959).

Let $f(x, \theta)$ be a probability density over the range $a(\theta) \leqq x \leqq b(\theta)$, where θ is a k-dimensional parameter. Let $T(x, \theta)$ be a function of x and θ. Assuming that the various derivatives and integrals exist, define $\delta(\theta) = E(\partial \log f/\partial x)$, $J(\theta) = -E(\partial^2 \log f/\partial x^2)$, and $\psi(\theta) = E(T)$. Using the usual techniques to prove the Cramér-Rao inequality the author shows that

$$\operatorname{Var}(T) \geqq \frac{\left\{T(b)f(b,\theta) - T(a)f(a,\theta) - E\left(\frac{\partial T}{\partial x}\right) - \psi(\theta)\delta(\theta)\right\}^2}{J(\theta) + \left.\frac{\partial f}{\partial x}\right|_{x=b} - \left.\frac{\partial f}{\partial x}\right|_{x=a} - \delta^2(\theta)}.$$

A multidimensional version of the inequality is also established.

J. R. Blum (Albuquerque, N.M.)

5255:

Gart, John J. An extension of the Cramér-Rao inequality. Ann. Math. Statist. **30** (1959), 367–380.

The extension of the title is obtained by integrating both sides of Schwarz's inequality with respect to an a priori distribution, with analogues in the multivariate, Bhattacharyya, and sequential treatments. For example, for the expectation of the variance of an unbiased estimator one obtains the lower bound $1/EI_k(\theta)$, where $I_k(\theta)$ is the "information" from k observations when θ is the true parameter value. (This is not as good as the

bound $E\{1/I_k(\theta)\}$ which one obtains on directly taking expectations of both sides of the classical inequality of the title.) *J. Kiefer* (Ithaca, N.Y.)

5256:

Wijsman, Robert A. Incomplete sufficient statistics and similar tests. Ann. Math. Statist. **29** (1958), 1028–1045.

The main result of this interesting paper is a method, called D-method, of constructing non-Neyman structure similar regions. The method is applicable to families of densities of a random variable X with a sample space W, depending upon a k dimensional parameter vector θ for which there exists an m dimensional statistic $T=(T_1, T_2, \cdots, T_m)$, with $m>k$. The additional conditions are that the probability density of T be of the exponential family $\{h(t)\exp\{\sum_1^m S_i(\theta)t_i\}$ with the coefficients $S_i(\theta)$ satisfying the identity $P\{S_1, S_2, \cdots, S_m\}\equiv 0$. Here P is a polynomial of order σ in the S_i with coefficients possibly also depending upon the S_i but not directly on θ. The function $h(t)$ must be such that there exists an m dimensional cube c defined by $a_i\leq t_i\leq b_i$, within which $h(t)$ is bounded away from zero.—Let D be a differential operator defined by $D\equiv P(\partial/\partial S_1, \partial/\partial S_2, \cdots, \partial/\partial S_m)$. Let $G(t)$ be a function with partial derivatives up to the order σ and vanishing identically outside of c. For example, inside of c, G may be given by $\prod_1^m (b_i-t_i)^\sigma(t_i-a_i)^\sigma$. Next, let $0<\alpha<1$ and k be such that, for all t, $0\leq\alpha+kDG/h\equiv\varphi(t)\leq 1$. Let w be a measurable subset of W such that, for all t, the conditional probability $P\{X\in w|T=t\}=\varphi(t)$. It is easy to verify that $P\{X\in w\}\equiv\alpha$ so that w is a similar region. However, if $\varphi(t)\not\equiv\alpha$, this region is not of Neyman structure.—Applications include one case where, apart from differences in terms of sets of probability zero, the family of regions obtainable by the D-method contains all possible similar regions. The paper is inspired by lectures of E. L. Lehmann. *J. Neyman* (Berkeley, Calif.)

5257:

Carlson, Phillip G. Tests of hypothesis on the exponential lower limit. Skand. Aktuarietidskr. **1958**, 47–54 (1959).

Let a sample of size n be drawn from the e.d.f. $F(x;\theta,A)$, where $F(x;\theta,A)=1-e^{-(x-A)/\theta}$, $x\geq A$, $=0$, elsewhere. Let the observations be arranged in order of size, $x_{(1)}\leq x_{(2)}\leq\cdots\leq x_{(n)}$. The author proposes to test hypotheses about A by using the statistic $h^*=(x_{(1)}-A)/(x_{(n)}-x_{(1)})$. The exact distribution of h^* is derived and upper .90, .95, and .99 percentage points are given for $n=2(1)\ 10$. The author's test, while easy to compute, has considerably less power than the best test which is based on the statistic $g^*=(x_{(1)}-A)/(\bar{X}-x_{(1)})$.

Benjamin Epstein (Stanford, Calif.)

5258:

Tukey, John W. A quick, compact, two-sample test to Duckworth's specifications. Technometrics **1** (1959), 31–48.

"A simple test for comparing two samples is proposed. If one sample contains the highest value and the other the lowest value of the two samples combined, the total of two counts is taken as the statistic. These counts are of (i) the number of individuals in the one sample above all individuals in the other, and (ii) the number of individuals in the other below all those in the one. 5 per cent, 1 per cent and 0.1 per cent critical values are roughly 7, 10 and 13, respectively. Precise values for pairs of small and moderate sized samples are given, as are asymptotic critical values. Derivations are included." (Author's summary) *F. J. Anscombe* (Chicago, Ill.)

5259:

Stone, M. Application of a measure of information to the design and comparison of regression experiments. Ann. Math. Statist. **30** (1959), 55–70.

The measure of information introduced by Lindley [same Ann. **27** (1956), 986–1005, MR **18**, 783], and which was shown by him possibly to lead to the choice of inadmissible experiments, is introduced in the usual linear regression theory by assuming the unknown parameter k-vector θ has an a priori Gaussian distribution with means 0 and covariance matrix A. The problem then reduces to choosing the design information matrix F to maximize $\det(I+AF)$. Since the design maximizing $\det(I+AF)$ depends on A, which will usually be unknown even if the Gaussian-Bayesian point of view is applicable, the approach of using A is mainly a mathematical device. For example, using the fact that $\det(I+AF)\geq\det(I+AG)$ for all positive definite A if and only if $F-G$ is non-negative definite, proofs of various results of Ehrenfeld [*Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, Vol. 1*, pp. 57–67, Univ. of Calif. Press, Berkeley-Los Angeles, 1956; MR **18**, 946] are obtained. Also, as $m\to\infty$, with $A=mI$, the integral with respect to the a priori normal distribution of the power function of the analysis variance test of the hypothesis $\theta=0$ is maximized by the design which maximizes $\det F$. (Actually, much more is true. Wald's theorem implies that the integral of the power function on every sphere about 0 is maximized by this design [see A. Wald, Ann. Math. Statist. **13** (1942), 434–439; MR **5**, 129].) Designs which maximize $\det F$ are determined in several simple situations which fall within the general considerations of the reviewer [ibid., **29** (1958), 675–699; MR **20** #4910] regarding this maximization in symmetric settings. *J. Kiefer* (Ithaca, N.Y.)

5260:

Graybill, Franklin A.; and Weeks, David L. Combining inter-block and intra-block information in balanced incomplete blocks. Ann. Math. Statist. **30** (1959), 799–805.

A slight modification of Yates' weighted estimates of the fixed treatment effects in a balanced incomplete block design with random block effects [H. Scheffé, *The Analysis of Variance*, John Wiley & Sons, New York, Chapman & Hall, Ltd., London, 1959] is shown to be unbiased and based on an incomplete minimal set of sufficient statistics in the Lehmann-Scheffé sense [Sankhyā **10** (1950), 305–340; MR **12**, 511].

H. L. Seal (New Haven, Conn.)

5261:

Dykstra, O., Jr. Partial duplication of factorial experiments. Technometrics **1** (1959), 63–75.

Two-level fractional factorial experiment designs are described where some treatment combinations are dupli-

cated. Experiment plans are included ranging from six to eleven factors. The analysis is outlined.

M. Zelen (Washington, D.C.)

5262:

Le Cam, Lucien. Les propriétés asymptotiques des solutions de Bayes. Publ. Inst. Statist. Univ. Paris **7** (1958), no. 3/4, 17–35.

In this paper the author investigates, within the framework of the Wald theory of decision rules, the asymptotic properties of Bayes solutions. Definitions and notation: Θ, an arbitrary set whose elements θ are the states of nature; $\mathfrak{X}$, the space of values of the observable random elements; Δ, the space of possible decisions; $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are 'tribes' (σ-algebras) of subsets of $\mathfrak{X}$, Θ and Δ, respectively; $W(\theta, t)$ a function defined on $\Theta\times\Delta$; $\mathscr{D}$, the set of permissible decision rules (a decision rule is a function $x\rightarrow F_x$ such that for all $x\in\mathfrak{X}$ there is associated a probability distribution F_x defined on $\mathscr{C}$, i.e., if x is observed the statistician chooses a point $t\in\Delta$ at random with measure F_x); P_θ is a probability measure defined on A for every $\theta\in\Theta$. Hypotheses: (1) For all $A\in\mathscr{A}$, the function $\theta\rightarrow P_\theta(A)$ is measurable with respect to $\mathscr{B}$; (2) $W(\theta, t)$ is measurable with respect to $\mathscr{B}\times\mathscr{C}$, with $W(\theta, t)\in[0, 1]$ for any $\theta\in\Theta$ and $t\in\Delta$; (3) the decision rules belonging to $\mathscr{D}$ are measurable in the sense that for all $C\in\mathscr{C}$ $x\rightarrow F_x(C)$ is measurable with respect to $\mathscr{A}$; (4) Let $T=(x\rightarrow F_x)$, $T'=(x\rightarrow F_x')$ be two elements of $\mathscr{D}$, and let v be a numerical function which is measurable with respect to $\mathscr{A}$. Then the decision rule $T''=(x\rightarrow F_x'')$, where $F_x''=([1-v(x)]F_x+v(x)F_x')\in\mathscr{D}$.

For $T\in\mathscr{D}$, let $W[\theta, T, x]=\int_\Delta W(\theta, t)F_x(dt)$, and let $R[\theta, T]=\int_{\mathfrak{X}}W[\theta, T, x]P_\theta(dx)$ denote the risk corresponding to T when the state of nature is θ. Also, let $\mathfrak{M}$ denote the space of bounded numerical functions, L the space of numerical functions defined on Θ that are equivalent for μ to functions measurable with respect to B and integrable with respect to μ, L_a the space of accessible functions, and L_a^+ the cone of positive elements of L_a. Finally, μ denotes the probability measure on $\mathscr{B}$ which plays the role of an a priori probability; and N denotes a directed set of indices.

The main result is the following theorem: Suppose all problems indexed by N satisfy hypotheses (1)–(4). Let $\{T_n\}$, $T_n\in\mathscr{D}_n$ be a family of decision rules that is an asymptotic Bayes solution with respect to μ. Let $\{T_n'\}$, $T_n'\in\mathscr{D}_n$ be an arbitrary family of decision rules indexed by N. Let $\{\varphi_n\}$ be a family of elements of $\mathfrak{M}$ defined by: $\varphi_n(\theta)\ \ R_n(\theta, T_n')-R_n(\theta, T_n)$. Then: (1) T_n is also an asymptotic solution for all $f\in L_a^+$; (2) There exist 'adherent' points to the directed system $\{\varphi_n, n\in N\}$ and all 'adherent' points are positive accessible; (3) In order that $\{T_n'\}$ also be an asymptotic Bayes solution for μ it is necessary and sufficient that for all $f\in L_a$ the integrals $\int\varphi_n f d\mu$ converge to zero.

Particular cases and applications are also discussed.

A. T. Bharucha-Reid (Eugene, Ore.)

5263:

Wesler, Oscar. Invariance theory and a modified minimax principle. Ann. Math. Statist. **30** (1959), 1–20.

The modified minimax principle of the title considers the supremum of the risk function on each orbit of the group, the invariant procedures being a complete class under suitable assumptions [minimax invariance theory for vector-valued risk was also considered by M. Peisakoff, Thesis, Princeton Univ., 1950]. The method of proof of the invariance theorem is an extension of that of Hunt and Stein [see E. L. Lehmann, Ann. Math. Statist. **21** (1950), 1–26; MR **11**, 528] to a more general setting, the minimax proof holding for each orbit; for example, Riesz's lemma is replaced by a proof that the space of decision functions is appropriately compact [for which see also A. Wald, *Statistical Decision Functions*, Wiley, New York, 1950; MR **12**, 193; or, in a very general setting, L. LeCam, Ann. Math. Statist. **26** (1955), 69–81; MR **16**, 730]. The condition on the group is that used by Hunt and Stein, Peisakoff, the reviewer [ibid. **28** (1957), 573–601; MR **19**, 1097] and others. The other assumptions are for the most part in the most general form yet to appear in print, although not in a form that can be verified in applications as readily as those of the above references. {Peisakoff and the reviewer do not require the author's assumption of σ-finite domination.}

J. Kiefer (Ithaca, N.Y.)

5264:

★Špaček, Antonín. An elementary experience problem. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 253–258. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

The author corrects an error in an earlier paper [Czechoslovak Math. J. **6** (1956), 190–194; MR **19**, 189] by making a slightly stronger assumption. The theory of experience originated by the author has been greatly developed by Winkelbauer (see next review).

K. J. Arrow (Santa Monica, Calif.)

5265:

★Winkelbauer, Karel. Experience in games of strategy and in statistical decision. Transactions of the first Prague conference on information theory, statistical decision functions, random processes held at Liblice near Prague from November 28 to 30, 1956, pp. 297–354. Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957. 354 pp. Kčs 34.00.

This is a considerable elaboration of the theory of experience introduced by Fabian and Spacek [Czechoslovak Math. J. **6** (1956), 190–194; MR **19**, 189, and preceding review]. He gives a very general formulation of the notion of learning from experience in successive plays with respect both to statistical decisions and to games of strategy. It is impossible to summarize the large variety of results, but the following is a rough paraphrase of the main type of result: if a statistician has a consistent estimate of the a priori distribution of a parameter and if he chooses a sequence of decisions which are asymptotically optimal for the sequence of estimates, then the sequence of average expected payoffs converges to the minimum expected payoff with full knowledge of the a priori distribution.

The discussion includes a number of theorems on the compactness and related properties of games and on various convergence properties of strategic procedures which may be useful in other contexts.

K. J. Arrow (Stanford, Calif.)

5266:

Allen, William R. Inference from tests with continuously increasing stress. Operations Res. **7** (1959), 303–312.

"This paper contains a brief résumé of a mathematical foundation for construction and analysis of accelerated life tests. The effects, on distribution of time to failure, of several special methods of accelerating life are studied. A special type of accelerated life test for capacitors is examined in detail. Three methods for estimating the parameter of the original distribution, from data gathered from the accelerated tests, are indicated if the original distribution is negative exponential." (Author's summary)

Benjamin Epstein (Stanford, Calif.)

5267:

Plackett, R. L. The analysis of life test data. Technometrics **1** (1959), 9–19.

n items are placed on life test. The test is discontinued after the first k ($\leq n$) items have failed. The author gives approximate, graphical, and exact methods for estimating the relevant parameters if life length is either normally distributed or follows a logistic distribution.

Benjamin Epstein (Stanford, Calif.)

NUMERICAL METHODS

See also 4962, 5056, 5300, 5373, 5415, 5515, 5526.

5268:

★Kantorovich, L. V.; and Krylov, V. I. Approximate methods of higher analysis. Translated from the 3rd Russian edition by C. D. Benster. Interscience Publishers, Inc., New York; P. Noordhoff Ltd., Groningen; 1958. xv+681 pp. $17.00.

This work is a translation from the third Russian edition [*Približennye metody vysšego analiza*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; MR **13**, 77], which was a minor revision of earlier editions. The work is concerned with what Lanczos calls 'parexic analysis', a study of processes which lead to approximate solutions of the problems of mathematical physics by rigorous methods. This is in contradistinction to 'numerical analysis' which deals with the translation of mathematical processes into numerical algorisms, and is greatly aided by the discoveries of parexic analysis. In other words, the work under review presents a survey of analytical tools which are of importance in numerical analysis, with emphasis on error estimation. In Chapter I methods based on the representation of solutions as infinite series, such as series expansion in orthogonal polynomials, are presented. Chapter II deals with Fredholm integral equations, Chapter III with the method of finite differences, Chapter IV with variational methods. In the latter, the methods of Ritz and Galerkin are examined in detail with careful considerations of convergence and error estimation. It is one of the most valuable chapters in the book. Chapters V, VI and VII deal with various approximate methods of conformal mapping which have not before been available in book form in the English language. The date of writing of the book is essentially 1941, with consequent limitations of completeness. Within this restriction, however, the work is invaluable for the applied mathematician interested in approximate solutions, and the translation is entirely adequate.

W. F. Freiberger (Providence, R.I.)

5269:

★Cashwell, E. D.; and Everett, C. J. A practical manual on the Monte Carlo method for random walk problems. International Tracts in Computer Science and Technology and their Application, Vol. 1. Pergamon Press, New York-London-Paris-Los Angeles, 1959. ix+153 pp. $6.00.

The Monte Carlo method usually implies that some device is used to generate "observations". Further, it is assumed that statistical sampling techniques can be employed to insure that these "observations" can be used to provide a statistical estimate to the solution of the given physical or analytical problem. The present authors are concerned with the interplay of neutrons and photons with bulk matter in geometric systems of varying complexity. Particles are assumed to emanate from a specified source. The type of problem under consideration is to estimate the percentage of particles that can be expected to terminate in stipulated categories. It is assumed that upon leaving the specified originating source the particles are subjected to specified actions in a material medium of known geometry. Here, the device for generating the "observations" and performing the subsequent analysis is a digital computer. Each "observation" consists of the life history of a particle which is "born" at a given source and "dies" upon reaching a terminal category.

The authors refer to their text as a practical manual on the Monte Carlo method for random walk problems. The editors and publishers acknowledge "the need for speedy publication in this rapidly developing field". They also apologize that "many workers in this fast moving field cannot devote the necessary time to producing a finished monograph". No apologies are needed for this manual. The authors and editors are to be congratulated on their fine work. They have set an excellent example for others to follow. Considering that the text is not a "finished monograph" the quality of printing is excellent. No major misprints were noted, however, the reviewer was not able to find Figure 59a.

By Chapters, the text is developed as follows. (1) "Basic principles"—A general outline of the procedures to follow including an explanation of how "pseudo random numbers" are utilized. (2) "The source routine"—How a problem is initiated by the machine, how "print-outs" are effected, and how the particles are drawn from the source. The next few chapters contain material specific to the behavior of particles in a given system. (3) "The mean free path and transmission". (4) "The collision or escape routine". (5) "The collision routine for neutrons". (6) "Photon collisions". (7) "Direction parameters offer collision". (8) "Terminal classification". (9) The last two chapters are: "Remarks on computation". (10) "Statistical considerations". With regard to chapter Nine it is strongly recommended that before one uses the procedure mentioned on page 118 for generating "pseudo random numbers", they consider other approaches such as mentioned, for example, Juncosa [Ballistics Res. Lab. Aberdeen Proving Ground Rep. No. 855 (1953); MR **15**, 559] and Taussky and Todd [*Symposium on Monte Carlo methods*, Univ. of Florida, 1954, pp. 15–18, Wiley, New York, 1956; MR **18**, 239]. The manual closes with an

excellent appendix which contains the summary of twenty problems which have been coded and run on the MANIAC 1 at Los Alamos.

The text is orientated to a specific class of problems. However, when it is used in conjunction with the material from this Symposium on Monte Carlo, specifically papers by Kahn [pp. 146–190; MR **18**, 151], Butler [pp. 249–264; MR **18**, 152], Trotter and Tukey [pp. 64–79; MR **18**, 152], Marshall [pp. 123–140; MR **18**, 153] and Walsh [pp. 265–277; MR **18**, 153; see also Wendel, Ann. Math. Statist. **28** (1957), 1048–1052; MR **20** #409], this material should offer an excellent survey for many of the techniques and problems which can be employed when undertaking other types of simulations on a digital computer.

M. Muller (New York, N.Y.)

5270:

Vionnet, Monique. Approximation de Tchebycheff d'ordre n des fonctions $\sin x$, $\sin x/x$, $\cos x$ **et** $\exp x$. Chiffres **2** (1959), 77–96. (English, German and Russian summaries)

The Fourier coefficients of $\operatorname{Sin} x$, $x^{-1} \operatorname{Sin} x$, $\operatorname{Cos} x$ and e^x in the orthogonal system of Tchebycheff polynomials $T_n(x) \operatorname{Cos}(n \operatorname{Arccos} x)$, $-1<x<1$, are computed with extreme accuracy (to 10^{-24}) up to $n=18$.

These numerical results are very important when, working with an electronic computer, a programmer has to compose an economical subroutine for functions listed above.

When applied to this problem, the author's coefficients should be first transformed into those of an equivalent polynomial in the variable x, argument of the function. A further transformation of this polynomial into the equivalent finite continued fraction would save again almost 50% of electronic computer's time.

E. G. Kogbetliantz (New York, N.Y.)

5271:

Şaichin, A. Sur l'application simultanée de la méthode de Newton et de la méthode des parties proportionnelles. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 331–337. (Romanian. Russian and French summaries)

5272:

Béla, Jankó. La méthode de Newton et la solution approximative des systèmes d'équations algébriques linéaires. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 103–114. (Romanian. Russian and French summaries)

"Dans ce travail on donne des méthodes nouvelles pour résoudre les systèmes linéaires finis, en généralisant les méthodes connues de Pollaczek—Geiringer et de Seidel.

Ici le système linéaire donné a été transformé dans un système non-linéaire équivalent, choisi convenablement, de manière suivante: en appliquant la méthode de Newton généralisée par L. V. Kantorovitch—nous imposons la condition que la derivée de l'operation P soit une matrice de forme simple, par ex. diagonale, triangulaire etc."

Résumé de l'auteur

5273:

Wilkinson, J. H. Stability of the reduction of a matrix to almost triangular and triangular forms by elementary similarity transformations. J. Assoc. Comput. Mach. **6** (1959), 336–359.

A method for the reduction of an arbitrary matrix to almost triangular (Hessenberg) form by means of similarity transformations involving only elementary (rational) row and column operations is described. Unitary transformations, as proposed by Givens [Oak Ridge Nat. Lab. Rep. **ORNL 1574** (1954); J. Assoc. Comput. Mach. **4** (1957), 298–307; MR **16**, 177; **19**, 1081] are customarily used for this purpose for reasons of numerical stability. Wilkinson shows that, with appropriate permutations of rows (and columns), the elementary transformations are equally accurate and involve much less computation.

A method proposed recently by Givens [J. Soc. Indust. Appl. Math. **6** (1958), 26–50; MR **19**, 1081] for the reduction of an arbitrary matrix to triangular form, with the computation of an additional eigenvector at each step, is discussed from the point of view of numerical stability. Wilkinson proposes a modified method, using an independent evaluation of each eigenvector to achieve stability, which involves only elementary row and column operations.

B. A. Chartres (Sydney)

5274:

★**Blanc, Charles. Calcul numérique et considérations stochastiques.** Symposium on modern computers: Proceedings of the Rome Symposium (17–18 October 1956) organized by the Preparatory Committee of the International Computation Center under the auspices of Unesco, pp. 23–26. Libreria Eredi Virgilio Veschi, Rome. vii+94 pp.

A brief discussion of the error involved in the use of an approximate integration formula when the integrand is a stationary stochastic process of order two.

T. N. E. Greville (Kensington, Md.)

5275:

Schechter, Ervin. Sur l'erreur du procédé d'intégration numérique de Runge-Kutta. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 115–124. (Romanian. Russian and French summaries)

"Dans le présent article on généralise, au cas d'un nombre quelconque de pas, une délimitation donnée par Bieberbach pour l'erreur de la méthode d'intégration numérique de Runge-Kutta, valable seulement pour le premier pas. On considère ensuite l'erreur d'arrondiment, en montrant sur un exemple simple que la méthode de Runge-Kutta, bien que convergente en ce qui concerne l'erreur du procédé, peut devenir divergente, si l'on considère aussi l'accumulation des erreurs d'arrondiment.

Au dernier paragraphe les résultats sont étendus aux systèmes d'arrondiment."

Résumé de l'auteur

5276:

Vejvoda, Otto. Die Fehlerabschätzung der Runge-Kutta-Formel. Apl. Mat. **2** (1957), 1–23. (Czech. Russian and German summaries)

Error estimates for the numerical solution of the initial value problem for a system of n first order nonlinear differential equations $y_i' = f_i(x, y_1, \cdots, y_n)$ $(i=1, \cdots, n)$

by the Runge-Kutta method are considered. Estimates for the upper bound of the error in the usual Runge-Kutta formula after one step are given using various assumptions regarding the magnitude of the functions, f_k, and of their partial derivatives up to a prescribed order. With $f_k \in C_5$ the expansions of the solutions may be carried out up to terms of order 6. These assumptions differ from those made by Bieberbach [Z. Angew. Math. Physik **2** (1951), 233–248; MR **13**, 286] and subsequent ones made by Max Lotkin [Math. Tables Aids Comput. **5** (1951), 128–133; MR **13**, 286] in that the magnitudes in question are estimated in terms of only one rather than two constants. The difficulties in doing this are pointed out. Estimates for the error after s steps and several examples and comparisons are presented. *J. A. Nohel* (Atlanta, Ga.)

5277:

Borwein, David; and Mitchell, Andrew R. The effect of boundary conditions and mesh size on the accuracy of finite difference solutions of two-point boundary problems. Z. Angew. Math. Phys. **10** (1959), 221–232. (German summary)

The numerical solution of two-point boundary problems in ordinary differential equations may be difficult whenever the range of solution is "critical", that is when either the differential equation or the associated difference equations have no solution. This situation is examined for a general linear second-order equation, with general boundary conditions involving both the function and its first derivative. The solution of the finite-difference equations is also examined in near-critical cases, and it is shown that the various residuals may have very different effects on the solution. In these circumstances, for which the matrix is nearly singular, the reviewer dissents from the recommendation that iterative methods are more valuable than direct methods. *L. Fox* (Oxford)

5278:

Keller, Herbert B. Approximate solutions of transport problems. I. Steady-state, elastic scattering in plane and spherical geometry. J. Soc. Indust. Appl. Math. **6** (1958), 452–465.

The author discusses the approximate solution of the steady state transport problem

$$\left\{\mu \frac{\partial}{\partial x}+\sigma(x, u)\right\}\Phi(x, \mu, u)-S[\Phi; \mu, u] = \mathscr{S}(x, \mu, u)$$

in an infinite plane slab of finite thickness. Here, $\Phi(x, \mu, u)$ is the neutron flux at position x, lethargy u, whose velocity vector makes an angle $\cos^{-1}\mu$ with the positive x axis; $\sigma(x, u)$ is the total macroscopic cross-section for neutrons at x with lethargy u; $\mathscr{S}(x, \mu, u)$ is the inhomogeneous source of neutrons, and $S[\Phi; \mu, u]$ is the term involving the elastic scattering of neutrons.

While others [e.g., B. Carlson, Los Alamos Sci. Lab. Rep. **LA-1891** (1955)] have approximated the solution of the above transport equation by differencing all the variables x, μ, and u, the author of this paper introduces only angular and lethargy differencing, and leaves the space variables x continuous. Thus, the neutron flux $\Phi(x, \mu, u)$ is approximated by $\Phi(x, \mu_j, u_n) = \Phi_j^n(x)$. With suitable approximations to the scattering term $S[\Phi; \mu, u]$ and the inhomogeneous source $\mathscr{S}(x, \mu, u)$, and system of differential-difference equations is obtained, whose closed solution is explicitly given. In this sense, the method is analogous to the method employed by Hartree and Womersley [Proc. Royal Soc. London Ser. A **161** (1937), 353–366] in the numerical solution of parabolic differential equations in the plane.

The author then shows the consistency (difference operator approaches differential operator) and convergence ($\Phi_j^n(x)$ approaches $\Phi(x, \mu, u)$) of the method as $\Delta\mu$ and Δu approach zero, as well as the stability (small changes in $\mathscr{S}(x, \mu, u)$ produce small changes in $\Phi_j^n(x)$) of the method. Finally, extensions, such as to the case of spherical geometry, are indicated. *R. S. Varga* (Cleveland, Ohio)

5279:

★Goto, Motinori. Numerical calculation of an eigenfunction of a partial differential equation whose coefficients are functions of the integral of the eigenfunction. Symposium on the numerical treatment of partial differential equations with real characteristics: Proceedings of the Rome Symposium (28–29–30 January 1959) organized by the Provisional International Computation Centre, pp. 117–130. Libreria Eredi Virgilio Veschi, Rome, 1959. xii+158 pp.

The author considers the numerical solution of the eigenvalue problem

$$(*)\qquad -\nabla^2\Phi+\frac{1}{M^2}\Phi = \frac{k_\infty}{M^2}\Phi$$

in cylindrical coordinates, which arises from the steady-state critical equation of a reactor, assuming one energy group of neutrons.

After approximating (*) by difference equations, the author uses Chebyshev polynomials to accelerate the convergence of the power method to find the largest (in modulus) eigenvalue k_∞ of the corresponding matrix eigenvalue problem. In this respect, the results are already known [G. Birkhoff and R. S. Varga, J. Soc. Indust. Appl. Math. **6** (1958), 354–377; MR **20** #7407] for the more general problem of several energy groups of neutrons, in heterogeneous reactors. *R. S. Varga* (Cleveland, Ohio)

5280:

Vil'ner, I. A. Topology and geometry of a space of imaginary anamorphosis. Uspehi Mat. Nauk **13** (1958), no. 4 (82), 173–178. (Russian)

The author gives a real interpretation of certain nomograms. The interpretation involves real projective geometry, admitting complex extensions. No proofs are given. *A. Shields* (New York, N.Y.)

5281:

Bal, Lascu. Nomogrammes à transparent orienté pour les équations à quatre et à cinq variables. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 169–176. (Romanian. Russian and French summaries)

5282:

Radó, Francisc. La meilleure transformation projective des échelles de nomogrammes à points alignés. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 161–168. (Romanian. Russian and French summaries)

5283:

Hurwitz, H., Jr.; Pfeiffer, R. A.; and Zweifel, P. F. Numerical quadrature of Fourier transform integrals. II. Math. Tables Aids Comput. **13** (1959), 87–90.

Hurwitz and Zweifel [Math. Tables Aids Comput. **10** (1956), 140–149; MR **18**, 337] gave a formula for calculating the sine and cosine transforms of a function φ by constructing an auxiliary function σ that involves values of φ at a set of equally spaced points and applying $(2N)$-point Gaussian quadrature to an integral involving σ. Here a $(2N+1)$-point quadrature formula is applied for the same purpose. In principle, either method involves evaluating φ at a different set of points for each argument at which the transform is to be calculated. The authors now show that a suitable fixed set of points will allow the transform to be calculated at a number of different arguments, after which other values of the transform can be obtained graphically.

R. P. Boas, Jr. (Evanston, Ill.)

5284:

Sengupta, J. M.; and Bhattacharya, Nikhilesh. Tables of random normal deviates. Sankhyā **20** (1958), 249–286.

In Sankhyā **1** (1934), 289–328, a table of 10,400 pseudo-normal deviates appeared. These deviates were obtained by a conversion of pseudo-random digits developed by L. H. C. Tippett [*Random sampling numbers*, Cambridge University Press, 1927]. The statistical tests originally applied to the table gave no indication of inadequacies in the table. Since then doubt was raised concerning whether errors had been made in converting the pseudo-random digits to normal deviates, for example, the material found on pages 70 and 147 in *Distribution sampling with high speed computers*, by D. Teichroew [Thesis, University of North Carolina, 1953]. The present authors have done the conversion of the random digits again to arrive at a new set of 10,400 pseudo-normal deviates. In addition to the table the paper contains a brief historical introduction and a review of the results of: (1) two procedures which were employed to test normality of the deviates, (2) a test of "randomness" of the deviates. The authors appear to be satisfied with the results of the statistical tests. It is a pleasure to note that though the present authors and editors were aware of the much larger table published in 1955 by Rand Corporation [*A million random digits with 100,000 normal deviates*, The Free Press, Glencoe, Ill.; MR **16**, 749], they still decided to provide a way of correcting the discrepancies in the 1934 table.

M. E. Muller (New York, N.Y.)

5285:

★Kuntzmann, J. Calcul de fonctions de Legendre. Symposium on modern computers: Proceedings of the Rome Symposium (17–18 October 1956) organized by the Preparatory Committee of the International Computation Center under the auspices of Unesco, pp. 1–3. Libreria Eredi Virgilio Veschi, Rome. vii + 94 pp.

The Computing Laboratory of the University of Grenoble announces the preparation of a table of Legendre functions of non-integral order. Specifically, the project involves the computation of $P_n^{(m)}(\cos\theta)$ for a range of parameters $\theta=0°(1°)\,179°$; $n=-0.5(0.1)10.0$, $m=0, 1, 2$, and to nine or ten decimal places. This brief note discusses the error accumulation in the use of the recurrence relation

$$(n+1)P_{n+1} = -nP_{n-1}+(2n+1)\cos\theta P_n.$$

L. Fox (Oxford)

COMPUTING MACHINES

See also 4893, 4917, 4918, 4943.

5286:

Fischer, W. L. Demonstrationsmodell eines logistischen Rechengeräts. Math. Naturwiss. Unterricht **12** (1959/60), 353–357.

5287:

Polozova, N. La multiplication des séries trigonométriques à l'aide de la calculatrice électronique. Byull. Inst. Teoret. Astr. **6**, 757–760 (1958). (Russian. French summary)

The product of two Fourier series can be written as another Fourier series. In this paper it is described how the coefficients of the last series can be calculated on the BESM electronic calculator. The process can be used in problems of celestial mechanics.

H. A. Lauwerier (Amsterdam)

5288:

★Ercoli, Paolo; e Vacca, Roberto. Aritmetica binaria a precisione multipla: Confronto fra la soluzione programmata e quella automatica in una calcolatrice seriale. Symposium on questions of numerical analysis: Proceedings of the Rome Symposium (30 June–1 July 1958) organized by the Provisional International Computation Centre, pp. 47–57. Libreria Eredi Virgilio Veschi, Rome. vii + 79 pp.

5289:

Munteanu, Emil. Programmation du calcul de la correction des roues dentées. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **9** (1958), 225–236. (Romanian. Russian and French summaries)

"Ce travail donne les programmes, pour le calculateur électronique CIFA-1, du calcul des coordonnées des familles de courbes qui interviennent dans le calcul de correction des roues dentées." *Du résumé de l'auteur*

5290:

★Yamashita, H. Programming of linear algebraic equation with large-scale automatic relay computer FACOM-128B. Symposium on questions of numerical analysis: Proceedings of the Rome Symposium (30 June–1 July 1958) organized by the Provisional International Computation Centre, pp. 4–19. Libreria Eredi Virgilio Veschi, Rome. vii + 79 pp.

5291:

Blair, Charles R. On computer transcription of manual Morse. J. Assoc. Comput. Mach. **6** (1959), 429–442.

A digital computer is used to simulate a device for

transcribing messages sent in Morse code by a human being. Experimentation is facilitated by the use of the computer since the logic of the transcription method may be altered by simply changing the computer program.

Although mechanically produced Morse may be easily transcribed with existing equipment, messages sent by human operators cannot because of the variation in length among the symbols transmitted. The intra-character spaces (spaces between dots and dashes within a letter or character) and the inter-character spaces exhibit so much fluctuation that they cannot be distinguished by any absolute discrimination scheme. Instead, a combined distribution is formed for all types of spaces. The dividing line between intra-character spaces and inter-character spaces is then placed so as to maximize a so-called "goodness of separation" statistic. This dividing line, which is chosen on the basis of the message being received, may be shifted during reception.

A few other refinements are included in the computer program to sharpen the distinction between intra-character spaces and inter-character spaces, but in no way is the redundancy of letters in English used by the computer. All comparisons between human and computer transcription were correspondingly made with "context-free" messages. *D. E. Muller* (Urbana, Ill.)

MECHANICS OF PARTICLES AND SYSTEMS

See also 5043, 5216, 5483.

5292:

★**Mercier, André. Analytical and canonical formalism in physics.** North-Holland Publishing Co., Amsterdam; Interscience Publishers, Inc., New York; 1959. viii+222 pp. $6.75.

This book is designed to show that much of the formalism of modern field theory is pre-quantal in origin. To this end, the Lagrangian and canonical formalisms are developed in detail, using a terminology that makes contact with the modern lore of physics. Sections are also included on the specific classical field theories. Some areas in which quantum mechanics is essential to a deep understanding, such as the two-valuedness of the spinor field, and the role of the discrete symmetry transformations, are treated very lightly. In general the book achieves its end of providing a very general formulation of the analytical formalism in classical physics. However, it leaves this reviewer with the feeling that no real point has been made. *H. W. Lewis* (Madison, Wis.)

5293:

★**Synge, John L.; and Griffith, Byron A. Principles of mechanics. 3rd ed.** McGraw-Hill Book Co., Inc., New York-Toronto-London, 1959. xvii+552 pp. $9.50.

The first edition (1942) of this well-known treatise was reviewed in MR **3**, 213. The changes in this third edition are substantial. The notation has been changed to conform with the usual practice of alliterative symbols, such as v for velocity, a for acceleration, etc. But the major change is the addition of an entirely new part (Part III) on general methods and relativity. This consists of four chapters: The equations of Lagrange and Hamilton; Hamiltonian methods; Theory of vibrations; The special theory of relativity. It was added largely for the benefit of the student of theoretical physics and it is certain that he as well as others will welcome this masterly presentation of Hamiltonian theory which eases the transition from classical to quantum mechanics. The authors have succeeded in making their work appealing to the more advanced reader without detracting from its usefulness as an introductory text.

5294:

Meyer zur Capellen, W. Die Beschleunigungsänderung. II. Mitteilung. Ing.-Arch. **27** (1959), 73–87.

Nachdem in I [Ing-Arch. **27** (1959), 53–65; MR **21** #987] die allgemeine Theorie entwickelt worden ist, folgen jetzt besondere Fälle und Anwendungen. Durch Tabellen gibt der Verfasser eine Übersicht der besonderen Lagen der bewegten Ebene und der Konfiguration der merkwürdigen Punkte, wozu jetzt auch der Ruckpol gehört. Es folgen Betrachtungen über Rädertriebe und verschiedene Arten von Kurbeltrieben mit Berücksichtigung der für die allgemeine Theorie erhaltenen Sätze über den Ruck. *O. Bottema* (Delft)

5295:

Marguerre, K. Über die Lagrangeschen Gleichungen der Kinetik und der Elastostatik. Ing.-Arch. **28** (1959), 199–207.

An Hand eines Beispiels wird eine kritische Diskussion der Lagrangeschen Gleichungen geboten, wobei die Matrix-Schreibweise angewandt wird. Erst wird der Fall der Statik, dann derjenige der Kinetik betrachtet und die mechanische Deutung der Multiplikatoren angegeben. Eine Modifikation des Beispiels gibt Anlass nach der "anderen Seite" vorzustossen: in der Statik ist die Zahl g der Stützkräfte gleich der Zahl n der möglichen Starrkörperbewegungen; in der Kinetik ist $g<n$, in der Elastostatik dagegen $g>n$. Es entsteht so eine gewisse Dualität zwischen den Grundgleichungen der Kinetik und der Elastostatik. Der Verfasser wird dazu geführt, drei Typen von Kräften zu unterscheiden: eingeprägte Kräfte, Stützreaktionen oder geometrische Reaktionskräfte und einen neuen Typ, kinetische Reaktionskräfte oder Wechselkräfte. Auf die Frage: "Gibt es Lagrangesche Gleichungen der Elastostatik?" ist darum die Antwort des Verfassers: "Dem Sprachgebrauch nach nicht, und wir wollten dies auch nicht vorschlagen". Aber es wird gezeigt, "dass zwischen den kinetischen und den elastostatischen Gleichungen eine Entsprechung besteht, die das Recht gibt, in der Elastostatik von Lagrange zu sprechen". *O. Bottema* (Delft)

5296:

Klimov, D. M. On the motion of a gyroscope with universal (Cardan's) suspension and a nonaxially placed rotor. Soviet Physics. Dokl. **124** (4) (1959), 80–82 (537–539 Dokl. Akad. Nauk SSSR).

Der Autor zeigt, daß ein in einem Kardansystem gelagerter Kreisel systematische azimutale Auswanderungen besitzt, wenn die Rotorachse nicht genau mit der Hauptträgheitsachse übereinstimmt. Die dynamische Umwucht des Rotors führt dann zu erzwungenen Schwingungen des Systems. Diese Schwingungen wiederum können stets dann zu Auswanderungserscheinungen führen, wenn die Ebenen beider Kardanringe nicht senkrecht aufeinander stehen. *K. Magnus* (Stuttgart)

5297:

Theodorsen, Theodore. Optimum path of an airplane—minimum time to climb. J. Aero/Space Sci. **26** (1959), 637–642.

Detailed formulation for machine computation is given for minimizing the Euler-Lagrange time integral set up on the conventional flight equations of motion and so restricted that only one Lagrange multiplier is necessary. Weight change effects due to fuel consumption are undoubtedly small as suggested and are to be bracketed by estimated times at constant initial and terminal weight values. Initial and terminal end conditions such as, for example, terminal horizontal range, velocity, etc., are not incorporated directly into the mathematical setting and must be treated by judicious trial and error computing. Other complications such as restraints on aircraft coefficients (maximum lift and thrust coefficients) and passage through the transonic flight regime are handled by go-no go checks and probably by bracketing techniques. Conventional simplified flight polars are used in the subsonic and supersonic regimes respectively. Unfortunately, the method apparently was not tried on unclassified flight data so that experimental checks on the method are not presented. It is indicated that generally optimization requires shallower angles in the lower half of the climb and steeper ones in the upper half and that the method is useful for flight ceiling estimates.

M. G. Scherberg (Dayton, Ohio)

5298:

Mettler, E. Stabilitätsfragen bei freien Schwingungen mechanischer Systeme. Ing.-Arch. **28** (1959), 213–228.

In der linearisierten Näherungstheorie der Systeme mit mehreren Freiheitsgraden lassen sich beliebige Schwingungserscheinungen stets durch Überlagerung von voneinander unabhängigen Normalschwingungen erklären. Der Autor zeigt nun, daß die linearisierte Behandlung dieser Probleme nicht mehr ausreicht, wenn bestimmte Relationen zwischen den Eigenfrequenzen bestehen. Auch wenn die exakten Differentialgleichungen als Lösung eine Bewegung in nur einem Freiheitsgrad zulassen, so kann diese Bewegung instabil sein und zur Aufschaukelung von Schwingungen in einem anderen Freiheitsgrad führen. Diese bisher meist auf Kopplungseffekte zwischen den Eigenschwingungen eines linearen Systems zurückgeführten Erscheinungen lassen sich auch ohne Annahme irgendeiner Kopplung aus der Theorie der Systeme mit periodischen Koeffizienten (Hillsche bzw. Mathieusche Systeme) erklären. Einige Beispiele werden angegeben.

K. Magnus (Stuttgart)

5299:

Laricheva, V. V. Nonlinear damping of the natural vibrations of systems of arbitrary order. J. Appl. Math. Mech. **22** (1958), 745–749 (536–538 Prikl. Mat. Meh.).

A nonlinear, discontinuous damping of a linear vibrational system of order $2n$ (1) $B\ddot{x}+Ax=0$, where $B^{-1}A\sim \operatorname{diag}(w_v^2)$, $w_v^2>0$ and constant, is studied. The solutions of the second-order differential equation (2) $\ddot{x}+w^2x=F(x,\dot{x})$, where $w^2>0$, are damped if $F(x,\dot{x})$ is of the following form:

(3) $F(x,\dot{x}) = w^2 f(\operatorname{sign}(x\dot{x}))x \quad (f(1)=0,\ 0<f(-1)<1).$

This type of damping, that is, nonlinear, discontinuous and homogeneous of degree zero in x and $\dot{x}$, is applied to system (1). The system (4) $B\ddot{x}+Ax=F(x,\dot{x})$ can be transformed by the assumptions made on A and B to the normal form (5) $\ddot{\xi}+\operatorname{diag}(w_v^2)\xi=F^+(\xi,\dot{\xi})$. Defining $F^+(\xi,\dot{\xi})$, in analogy with (3), as

$$F^+(\xi,\dot{\xi}) = \operatorname{diag}(w_v^2 f_v(\operatorname{sign}(\xi_v,\dot{\xi}_v)))\xi \qquad (f_v(1)=0,\ 0<f_v(-1)<1),$$

where $v=1,2,\cdots,n$, the solutions of (5) are damped and are therefore those of (4). The form $F(x,\dot{x})$ takes on in (4) with the transformation-matrix T is

(6) $F(x,\dot{x}) = \operatorname{diag}(w_v^2 f_v(\operatorname{sign}((T^{-1}x)_v(T^{-1}\dot{x})_v)))Bx.$

Formula (6) is obtained and given in explicit form for the components of x and for $n=2$ only. A specialisation to a fourth-order differential equation is discussed finally.

K. Matthies (Columbia, S.C.)

5300:

Skowroński, Janisław. A method of qualitative analysis of vibrating discrete systems with strong non-linearity in the phase space. Arch. Mech. Stos. **10** (1958), 715–726. (Polish and Russian summaries)

In 1952, L. S. Jacobsen [J. Appl. Mech. **19** (1952), 543–553; MR **14**, 502] introduced the δ-method for solving numerically a second order autonomous differential equation by making small displacements in the phase plane. Recently this method has been investigated in more detail and extended to systems of second order equations by J. Skowroński and S. Ziemba [Zastosowanie metody delta do badania silnie nielniowych mechanicznych ukladow drgajacych o skonezonej ilośei stopni swobody, Biul. Wojskowej Akad. Tech. (1958)]. The present paper is concerned with a further extension of this method to nonautonomous systems of equations by letting the time variable assume only a discrete set of values.

J. K. Hale (Baltimore, Md.)

5301:

Novosyolov, V. S. Equations of motion of non-linear non-holonomic systems with variable masses. Vestnik Leningrad. Univ. **14** (1959), no. 7, 112–117. (Russian. English summary)

The well-known fundamental equations of motion, such as those containing undetermined multipliers, and those of the types of Čaplygin and Voronec-Hamel, as well as the Gauss principle of least constraint and the equations of Appell type, are extended to mechanics of variable mass. It is assumed that the motion of a system of variable mass is subject to non-linear non-holonomic constraints of the Četaev type, and that the latter do not depend on the change of the mass of the system in the course of time. [See also V. S. Novoselov, Leningrad Gos. Univ. Uč. Zap. **217** Ser. Mat. Nauk **31** (1957), 28–49, 50–83; Vestnik Leningrad Univ. **11** (1956), no. 19, 100–113; **12** (1957), no. 1, 130–140; MR **19**, 994, 898, 190, 695].

E. Leimanis (Vancouver, B.C.)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 5244, 5278, 5279, 5354, 5355, 5398, 5403, 5448, 5463, 5465, 5466.

5302:

★Ehrenfest, Paul; and Ehrenfest, Tatiana. The conceptual foundations of the statistical approach in

mechanics. Translated by M. J. Moravcsik. Cornell University Press, Ithaca, N.Y., 1959. xvi+114 pp.

This is the first English translation of the celebrated encyclopedia article of P. and T. Ehrenfest, published in 1912. It consists of a critical discussion of the foundations of (classical) statistical mechanics, in particular, the use of the concept 'probability', Boltzmann's *H*-theorem, the objections of Loschmidt and Zermelo and the various attempts to overcome them, and the difference between the approaches of Boltzmann and Gibbs. Since the theory of irreversible processes has become an important branch of statistical mechanics, this discussion has gained new interest. Many English-speaking readers will be able to profit from the critical attitude of the authors and the thorough way in which they recognize and analyze the difficulties, rather than hide them in some streamlined formalism. The translator has carried off an almost impossible task satisfactorily. {The footnotes, which form such an indispensable part of the text, are unfortunately moved to the end of the book, and the pregnant way in which Ehrenfest used italics has not been reproduced in the present version.} *N. G. van Kampen* (Utrecht)

5303:

Bergmann, Peter G.; and Morris, Eric. New potential for irreversible processes. Ann. Physics 8 (1959), 266–270.

The authors study a class of functionals of probability distributions in phase space satisfying the Liouville equation with certain random collision terms. They construct first functionals of two distributions which are positive definite, decreasing in time, and zero only when the two distributions are the same. Then, by taking the two distributions to have evolved one from the other over a (short) time, they develop functionals of a single distribution which are also positive definite, decreasing in time, and zero only when the distribution is stationary.

The authors announce their intention to use this latter class of functionals to approximate the stationary states of systems of physical interest.

S. Katz (New York, N.Y.)

5304:

Tchen, C. M. Kinetic equation for a plasma with unsteady correlations. Phys. Rev. (2) **114** (1959), 394–411.

Equations of motion for the single particle and pair distribution functions in a plasma are derived from the Liouville equation in the usual way. The system of equations is closed by replacing the three particle distribution function by a relation involving single particle and pair functions which is valid for weak interactions. The resulting equations contain screening and short distance correlation terms. An approximate expression for the Fourier transform of the pair functions is obtained and substituted into the equation for the single particle function.

The latter is simplified by assuming everything to vary slowly with time, leading to a Fokker-Planck equation for the diffusion of the single particle distribution function in momentum space. The coefficients are calculated by using the Maxwell distribution. *A. Herzenberg* (Manchester)

5305:

Trlifaj, Ladislav. On the relation between the method of spherical harmonics and the method of discrete coordinates. Czechoslovak J. Phys. **9** (1959), 535–543. (Russian. English summary)

"The relation between the method of spherical harmonics and the method of discrete coordinates is studied from the point of view of the integral Boltzmann equation for neutron diffusion problems which have full plane and cylindrical symmetry." *Author's summary*

5306:

Wahl, Philippe. Étude du mouvement Brownien de rotation de macromolécules en chaîne. J. Phys. Radium (8) **20** (1959), 680–682. (English summary)

On étudie le mouvement brownien de rotation d'une molécule formée de n chaînons rigides librement articulés les uns aux autres, la matière étant supposée localisée aux extrémités de chaque chaînon. On donne les équations des petits mouvements pour les déplacements plans, ainsi qu'une méthode de calcul des carrés moyens des déplacements angulaires de chaque chaînon pendant un petit intervalle de temps; les résultats sont explicités dans le cas d'une configuration à 2 ou 3 chaînons.

J. Naze (Marseille)

5307:

Abe, Ryuzo. Giant cluster expansion theory and its application to high temperature plasma. Progr. Theoret. Phys. **22** (1959), 213–226.

"The conventional virial expansion of thermodynamic functions is converted into a new expansion scheme, similar to the former but more powerful. The new method is particularly suitable to dealing with the interaction of long-range character, such as Coulomb potential, since it suffers from no divergence difficulties contrary to the conventional one. As an application of the method, the equilibrium properties of high temperature plasma are studied and the term of next higher order than the Debye-Hückel limiting law is obtained exactly. The order estimation indicates that the Debye-Hückel law is accurate within the error of a few per cent in this case. A possible extension of the present method to the theory of non-equilibrium properties of plasma or to quantum statistics is suggested." *Author's summary*

5308:

Chen, Chun-Sian. A new method in statistical perturbation theory. Soviet Physics. JETP **35** (8) (1959), 1062–1064 (1518–1521 Ž. Eksper. Teoret. Fiz.).

"A new variant of the statistical perturbation theory is presented, which consists in the setting up of a model of a dynamical system whose Hamiltonian contains the inverse temperature in parametric form. It is shown that for this dynamical system the perturbation theory is equivalent to the ordinary statistical perturbation theory and is much more convenient for practical calculation." (Author's summary) *S. Katz* (New York, N.Y.)

5309:

Bocchieri, P.; and Loinger, A. Ergodic foundation of quantum statistical mechanics. Phys. Rev. (2) **114** (1959), 948–951.

It is proved that for the "overwhelming majority" of

the initial states of a system the distribution laws of quantum statistical mechanics hold at the "overwhelming majority" of time instants. This is an ergodic theorem in a generalized sense, as strict ergodicity involves more stringent conditions about the equality of time and ensemble averages; the latter is not proved.
D. ter Haar (Oxford)

5310:

Engelmann, Folker. Über ein Theorem von Bloch. Z. Physik **155** (1959), 275–280.

It is shown that Bloch's theorem that the state of lowest free energy corresponds to zero current is a direct consequence of the symmetry of the Hamiltonian against time reversal. The consequences of this and the limitations to Bloch's theorem are discussed. *D. ter Haar* (Oxford)

5311:

Lebowitz, J. L. Stationary nonequilibrium Gibbsian ensembles. Phys. Rev. (2) **114** (1959), 1192–1202.

A Gibbsian statistical mechanics is developed for systems which are not necessarily close to equilibrium, though the theory is restricted to steady state situations. Systems are considered which are in contact with large heat reservoirs. A generalized Liouville equation is given and solved for some simple cases corresponding to systems in which heat conduction takes place. It is shown that a stationary non-equilibrium ensemble density is approached in a manner analogous to the approach to equilibrium in the usual ensemble theory. This stationary ensemble density bears a close resemblance to the usual canonical distributions. *D. ter Haar* (Oxford)

5312:

Kauman, W. G. Aspects of the phenomenological theory of diffusion. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 108–115.

The phenomenological diffusion equations may be written either so as to give the fluxes relative to some mean motion as a linear combination of the chemical potential gradients [L. Onsager and R. M. Fuoss, J. Phys. Chem. **36** (1932), 2689–2778], or to give the chemical potential gradients as linear combinations of velocities of the various components relative to each other [O. Lamm, Acta Chem. Scand., **11** (1957), 362–364]. The author derives relations between the two sets of phenomenological coefficients arising from these two formulations.
S. Prager (Minneapolis, Minn.)

5313:

Mazur, P. On the theory of Brownian motion. Physica **25** (1959), 149–162.

In recent papers Prigogine and Balescu [Physica **23** (1957), 555–568; MR **19**, 592] and Prigogine and Philippot [Physica **23** (1957), 569–584; MR **19**, 592] have derived from the Liouville equation a Fokker-Planck equation for a harmonic oscillator weakly coupled to a large number of other harmonic oscillators, assuming the latter to be in thermal equilibrium. They use the energy and angle variables for the oscillator which is outside equilibrium and find that it obeys a Markoff process of non-gaussian type (i.e. the conditional probability distributions are not gaussian). The present author considers various cases of Brownian motion and shows that by changing from cartesian to energy and angle coordinates the familiar equations of Brownian motion, which give a Markoff process of gaussian type, reduce to the equations of Prigogine and collaborators. Conversely the latter equations can be made gaussian by a simple coordinate transformation. The cases of Brownian motion considered are: a free particle in two dimensions, a charged particle in a magnetic field in two dimensions, and a one-dimensional harmonic oscillator in the case of small friction.
L. Van Hove (Utrecht)

5314:

★**Marquet, Simone. Base de la théorie cinétique des gaz équation de Boltzmann.** Le calcul des probabilités et ses applications. Paris, 15–20 Juillet 1958, pp. 123–132. Colloques Internationaux Centre National de la Recherche Scientifique, LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

This paper is devoted to a mathematical investigation of what is often called the master equation of a gas, i.e., the stochastic equation describing the effect of random collisions on the joint probability distribution of all particles of the gas. The methods of measure theory are used. The existence and uniqueness of the solution for given initial condition is established under suitable assumptions. It is further shown that if at some instant the joint probability distribution factorizes in a product of one particle distributions (molecular chaos property of Boltzmann) the same property holds at later times. The single particle distribution then verifies the Boltzmann equation. *L. Van Hove* (Utrecht)

ELASTICITY, PLASTICITY

See also 5295.

5315:

★**Landau, L. D.; and Lifshitz, E. M. Theory of elasticity.** Course of Theoretical Physics, Vol. 7. Translated by J. B. Sykes and W. H. Reid. Pergamon Press, London-Paris-Frankfurt; Addison-Wesley Publishing Co., Inc., Reading, Mass.; 1959. viii+134 pp. $6.50.

Translation of the second part of the authors' *Mehanika splošnyh sred*, 2d ed., Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953, reviewed in MR **16**, 412.

5316:

Graiff, Franca. Sulle condizioni di congruenza per deformazioni anche finite. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 415–422, 519–526.

These notes summarize practically everything known concerning the conditions of compatibility and add some further results. The author begins by considering finite strain. Using certain identities satisfied by the difference of Riemann tensors based upon two different metrics, she obtains conditions of integrability to be satisfied by a displacement vector in n-space. These are so complicated that she does not write out all the terms, but a simple form results in the classical case when one of the spaces is Euclidean n-space.

From this point on, the author considers the case of small strain. She begins by deriving by linearization of her results for finite strain the identities which had been obtained previously by Palatini [same Atti (6) **19** (1934), 466–469], who used the alternative formulation of the problem: to find conditions of integrability for the system of $2s_{(k|m)}=d$ in a given Riemannian space. In these identities appear terms containing the vector s_k and also the spin, $s_{(k|m)}$. These quantities are to be eliminated in order to obtain explicit conditions upon the stretching d_{km} alone. For spaces of constant curvature, the troublesome terms drop out. There is also some simplification when $n=2$, but even here the fully explicit elimination yields intricate results, as shown by the author's earlier paper [#5317]. The author sketches a treatment for general n, based upon variation of the independent invariants which can be formed from the Riemann tensor and its derivatives. Results in different spaces will differ from one another insofar as the number and nature of these invariants differ. The "general" case will be that in which the proper numbers of the Ricci tensor are distinct and non-constant; however, the more familiar special spaces all involve some degeneracy of the Ricci tensor. The author outlines a classification of the cases $n=3$ and $n=4$, determining the order of the resulting conditions, but she does not attempt explicit calculation.

C. Truesdell (Bloomington, Ind.)

5317:

Graiff, Franca. Sulle condizioni di congruenza per una membrana. Ist. Lombardo Accad. Sci. Lett. Rend. A **92** (1957/58), 34–42.

The problem of finding conditions of compatibility for the system $\xi_{(i,k)}=s_{(i,k)}$ is a problem in affinely connected spaces, and it should be possible to solve it when the connection is arbitrary. Only differential elimination is involved. Even for curved Riemannian spaces, however, the difficulty of the problem is notorious, and there is little literature concerning it. For the case when $n=2$, B. Finzi [same Rend. (2) **63** (1930), 975–982] found in general three equations of fourth order. He found also that a single equation of third order suffices for a surface applicable upon a surface of revolution, while for a surface of constant curvature there is a simple generalization of the classical condition of second order for the plane. On the basis of an indirect argument, the reviewer [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 1070–1072; MR **20** #2919] indicated that for a general surface, Finzi's system must be equivalent to a single equation of fifth order. The author attacks the problem more systematically by considering from the start a division into cases according as K and $\Delta^2 K$ are functionally independent or not. By a method of infinitesimal variation, the author obtains not only Finzi's result but also two identities connecting his three conditions. However, she does not carry out an explicit reduction to a single equation. [For discussion of related literature, see the later paper by the reviewer, Arch. Rational Mech. Anal. **4** (1959), 1–29.]

C. Truesdell (Bloomington, Ind.)

5318:

Kröner, Ekkehart; und Seeger, Alfred. Nicht-lineare Elastizitätstheorie der Versetzungen und Eigenspannungen. Arch. Rational Mech. Anal. **3** (1959), 97–119.

This is an extensive study of the theory of dislocations in elastic materials susceptible to finite deformations. First, the authors explain how a continuous distribution of incompatible local natural states in a body can be described by an affine connection. The symmetric part of this connection is related to the strain and the anti-symmetric part to the dislocation density. The basic equations of the theory express the requirement that the Einstein tensor of the connection should vanish in the deformed state. These basic equations together with the equilibrium conditions and an arbitrary elastic stress-strain law form a complete set of equations for the determination of the initial stress distribution corresponding to a prescribed dislocation density.

The authors describe an iterative method of solution, based on the use of stress functions. They elaborate their method in the special case of plane stress for the quadratic stress-strain law discussed by Murnaghan [*Finite deformations of an elastic solid*, Wiley, New York, 1951; MR **13**, 600]. They obtain explicit solutions for screw and step dislocations.

W. Noll (Pittsburgh, Pa.)

5319:

Spencer, A. J. M. On finite elastic deformations with a perturbed strain-energy function. Quart. J. Mech. Appl. Math. **12** (1959), 129–145.

The author investigates finite deformation of elastic material with strain energy of the form $W+\varepsilon W'$ where W is a strain energy function for which an explicit solution can be found and $\varepsilon W'$ is a perturbation. Application is made to the problem of simultaneous extension, inflation, and shear of a cylindrical tube, letting W be of Mooney form and letting W' be an arbitrary function of the strain invariants.

C. E. Pearson (Cambridge, Mass.)

5320:

Koiter, W. T. An infinite row of collinear cracks in an infinite elastic sheet. Ing.-Arch. **28** (1959), 168–172.

Muskhelishvili's solution for the stresses in a plane sheet weakened by a finite number of collinear cuts is extended to the case of an infinite number of cuts, which represents a limiting case of the effect of an infinite row of rectangular holes in an infinite sheet—a problem which is itself related to the problem of calculating shear flexibility in a box beam whose shear webs have rectangular cutouts. The solution is examined from several viewpoints, and numerical results for the strain energy associated with the cracks are plotted.

C. E. Pearson (Cambridge, Mass.)

5321:

Franciosi, Vincenzo. Osservazioni in tema di stati tensionali piani. Rend. Accad. Sci. Fis. Mat. Napoli (4) **25** (1958), 9–15.

The author has rediscovered that an exact state of plane stress in classical elasticity can only occur in certain special cases.

D. R. Bland (Manchester)

5322:

Legendre, Robert. Répartition des contraintes dans un anneau circulaire. C. R. Acad. Sci. Paris **249** (1959), 1084–1086.

"La répartition est calculée pour des efforts sur les contours représentables par des séries de Fourier."
Résumé de l'auteur

5323:

Legendre, Robert. Répartition des contraintes dans un disque. C. R. Acad. Sci. Paris **249** (1959), 945–946.

"La solution exacte du problème, définie par une fonction biharmonique complexe, est présentée."
Résumé de l'auteur

5324:

Chattarji, P. P. Stress concentrations around a small inclusion on the axis of a circular cylinder under torsion. Indian J. Theoret. Phys. **6** (1958), 51–64.

5325:

Skowroński, Janisław; and Ziemba, Stefan. Certain properties of mechanical models of structures. Arch. Mech. Stos. **11** (1959), 193–209. (Polish and Russian summaries)

"The article examines a model of a structure consisting of a discrete system in three-dimensional space of material points connected with one another by a damper and a spring both showing strong non-linearity. This set of points represents the mass distribution of the structure and the links between points represent the forces acting on the points according to the subdivision of the forces into internal and external forces. Typical boundary conditions of the structure are replaced by corresponding conditions for the coordinates of the fixed points of the model.

For the model under consideration, equations of motion are obtained in Cartesian coordinates in the three-dimensional space of the model and also in the configuration space. The potential of the elastic forces is determined, and also the general function of the dissipative and conservative forces. For specific boundary conditions of the model methods are considered for choosing generalized coordinates for the system. The distribution of the generalized forces into dissipative and conservative is analyzed. Conditions are given for the existence of a generalized Lagrange function and a generalized dissipation function. It is shown that from the character of the dissipative forces acting on the system it follows that there is positive dissipation of energy in the system. The equations of motion of the system are given in dynamical form, in configuration space for the generalized coordinates, and also in phase space, where an interpretation is given for the positive dissipation of the system." (Authors' summary)
L. S. D. Morley (Farnborough)

5326:

Ufliand, Ia. S. Mixed boundary value problem for an elastic lamina. Soviet Physics. Dokl. **123** (3) (1958), 1297–1299 (991–993 Dokl. Akad. Nauk SSSR).

A solution is obtained for the general three-dimensional problem of an isotropic elastic layer, on one of whose bounding planes the displacement components are specified and on the other the components of the normal stress vector, by expressing the Papkovich-Neuber harmonic functions as Hankel integrals. As an example these functions are determined for an elastic layer, one of whose faces is displacement free, the other being stress free except for a tangential force acting at one point.
W. D. Collins (Newcastle-upon-Tyne)

5327:

Shiriaev, E. A. Torsion of a circular bar with two cuts. J. Appl. Math. Mech. **22** (1958), 768–773 (549–553 Prikl. Mat. Meh.).

The conformal mapping

$$z = \left(\frac{d}{a}\right)^{1/2} \frac{1+2a\zeta+\zeta^2-b(1+2e\zeta^2+\zeta^4)^{1/2}}{1+2d\zeta+\zeta^2}$$

is used to get the torsion solution of a circular bar with two cuts along a diameter.

Stresses are calculated on the boundary. For the particular case of equal cuts the torsional rigidity is obtained and the results compared with those obtained by Shepherd [Proc. Roy. Soc. London Ser. A **138** (1938), 607–634].
B. R. Seth (Kharagpur)

5328:

Chattarji, P. P. Torsion of composite epitrochoidal sections. Z. Angew. Math. Mech. **39** (1959), 135–138. (German, French and Russian summaries)

"In this paper the techniques of complex function theory are adopted to solve the torsion problem of concentric composite elastic epitrochoidal sections of two different isotropic materials. The results have been obtained in closed forms." (Author's summary)
R. M. Morris (Cardiff)

5329:

Bassali, W. A. The transverse flexure of uniformly loaded and fully restrained thin isotropic plates with curvilinear edges. J. Mech. Phys. Solids **7** (1959), 145–156.

Continuing his work on the bending of thin plates [Proc. Cambridge Philos. Soc. **54** (1958), 265–287; MR **20** #3676], the author uses the complex function method to obtain exact solutions in finite terms for uniformly loaded regular curvilinear polygonal plates. The mapping function

$$z = c\zeta/(1+m\zeta^n)$$

is used. A number of particular cases, including that of an inverse of an ellipse are obtained and the results compared with those already known.
B. R. Seth (Kharagpur)

5330:

Mirsky, I.; and Herrmann, G. Axially symmetric motions of thick cylindrical shells. J. Appl. Mech. **25** (1958), 97–102.

5331:

Zaid, Melvin; and Paul, Burton. Oblique perforation of a thin plate by a truncated conical projectile. J. Franklin Inst. **268** (1959), 24–45.

In previous papers [same J. **264** (1957), 117–126; **265** (1958), 317–335; MR **19**, 595] the authors developed a simple momentum model, which corresponds to the penetration of a thin plate, as far as most of the important

effects are concerned. In this paper the method is extended as the title indicates.

J. W. Craggs (Newcastle-upon-Tyne)

5332:

Adadurov, R. A. An axially symmetric state of stress in a thin annular plate. Soviet Physics. Dokl. **124** (4) (1959), 221–224 (1005–1008 Dokl. Akad. Nauk SSSR).

The state of stress is investigated in an annular plate, attached at its inner and outer radii to rigid rings, and carrying moments applied to the rings in their planes. The plate is thin and incapable of carrying compressive stresses.

The linear formulation of the problem is solved.

H. D. Conway (Ithaca, N.Y.)

5333:

Tamate, O. Transverse flexure of a thin plate containing two circular holes. J. Appl. Mech. **26** (1959), 55–60.

"The problem of finding stress resultants in a thin elastic plate containing two circular holes of equal size, under plain bending about the axes of symmetry, has been discussed on the basis of the Poisson-Kirchoff theory. A method of perturbation is adopted for the determination of parametric coefficients involved in the solution. The factors of stress concentration are calculated and compared with the results available." *Author's summary*

5334:

Chovrebadze, D. S. The first boundary problem for a symmetrical shell of a particular shape. Soobšč. Akad. Nauk Gruzin. SSR **22** (1959), 137–144. (Russian)

Methods of shell analysis are used to calculate the deflections of a shallow doubly concave lens. A formulation of the basic equations due to Vekua is used whereby the problem reduces to the analysis of a two-dimensional continuum. The solution is assumed to have the form of a power series expansion in terms of a suitably small geometric parameter and the governing equations are ultimately treated by the complex methods of Mushcheli-shvili. It is seen that a treatment of this problem by conventional plate theory would introduce about 40% errors in the deflections for a typical example.

F. T. Geyling (Summit, N.J.)

5335:

Reissner, Eric. The edge effect in symmetric bending of shallow shells of revolution. Comm. Pure Appl. Math. **12** (1959), 385–398.

This paper is concerned with the existence and explicit treatment of boundary layer effects in rotationally symmetric (nonlinear) bending of isotropic shallow shells of revolution of uniform thickness. The surface loads are assumed to be in the direction of the axis of revolution. The author has previously shown [*Proceedings of symposia on applied math., vol. 3*, pp. 27–52, McGraw-Hill, New York, 1950; MR **12**, 557] that the problem can be reduced to two simultaneous ordinary differential equations of second order for a stress function ψ and an angular deflection β. Denoting the classical linear membrane solution by ψ_L, β_L and writing $\psi=\psi_L+\psi_S$, $\beta=\beta_L+\beta_S$ the author investigates under which conditions the supplementary solution ψ_S, β_S takes on significant values only in a narrow edge zone. This is done by introducing appropriate new variables and parameters, essentially a load parameter and a shell geometry parameter. The result is that for certain well defined ranges of these parameters the occurrence of a boundary layer can be deduced. The equations describing the behavior in the boundary layer turn out to be linear and can be solved explicitly.

The analysis is first carried out for non-linear membrane theory and contains as a special case the result of Bromberg and Stokes [Quart. Appl. Math. **3** (1945), 246–265; MR **7**, 142] for the uniformly loaded spherical shell. In the general case, where bending is significant, two ranges of the parameters are found for which a boundary layer exists. The first range, which physically corresponds essentially to the case of moderately large loads and sufficiently thin shells, is such that it contains a well defined subrange for which the equations for ψ_S, β_S reduce to those of the linear bending theory. In the second range, in which both load and geometry parameter may become large, the boundary layer equations are such that in a certain subrange they themselves are of the boundary layer type. This subrange corresponds to the case of non-linear membrane theory. Hence one has a secondary boundary layer within the nonlinear membrane boundary layer. The solution for the second range of parameters is carried out explicitly; in particular, the order of magnitude of the bending stresses in the secondary boundary layer is estimated. These stresses become significant for a built-in edge, in accordance with the fact that the boundary conditions of the nonlinear membrane theory cannot be satisfied in this case.

H. J. Weinitschke (Los Angeles, Calif.)

5336:

Rüdiger, D. Zur Theorie elastischer Schalen. Ing.-Arch. **28** (1959), 281–288.

Die übliche Theorie elastischer Schalen geht von der Voraussetzung aus, daß alle Punkte, die vor der Formänderung auf einer Normalen zur Schalenmittelfläche liegen, sich auch nach dieser auf einer Normalen zur verformten Mittelfläche befinden (Bernoullische Hypothese). Es ist wiederholt vermutet worden, daß diese Voraussetzung zu einer Inkonsequenz in den Grundgleichungen der Schalentheorie führt. Es treten nämlich im Elastizitätsgesetz Glieder auf, die von derselben Größenordnung sind, wie die vernachlässigte Querkraftverformung. In der vorliegenden Arbeit wird versucht, eine allgemeine Theorie der Schalen unter Berücksichtigung der Querkraftverformungen zu entwickeln und den Nachweis zu führen, daß diese Inkonsequenz in der bekannten Theorie tatsächlich besteht. Unter Verwendung der aufgestellten Theorie werden dann alle Terme von der Größenordnung der Querkraftverformung konsequent vernachlässigt und ein System von Schalengleichungen aufgestellt, das in sich widerspruchsfrei zu sein scheint.

W. Zerna (Hannover)

5337:

Zerna, W. Berechnung des Membranspannungszustandes doppelt gekrümmter Schalen über beliebigem Grundriss. Ing.-Arch. **28** (1959), 363–365.

The paper is concerned with the membrane state of stress in a "doubly-curved" roof with a base of an arbitrary shape which is symmetric about an axis. The problem is characterized by a single second-order partial differential

equation in terms of a stress function. Solution of the differential equation is not discussed.
P. M. Naghdi (Berkeley, Calif.)

5338:

Kączkowski, Zbigniew. Rectangular orthotropic thin plates with arbitrary boundary conditions. Arch. Mech. Stos. **10** (1958), 525–549. (Polish and Russian summaries)

A displacement function is obtained in the form of a double trigonometric series when the load is in the same form and the boundary is simply supported. For free supports along two opposite edges and also along three edges the displacement functions are expressed in single trigonometric series. It is shown that the limitation of the vanishing of the resultant force of the normal loads along the four edges can be removed where one displacement component vanishes and the other becomes independent of one variable.

For statical load and forced vibration an infinite system of nonhomogeneous linear equations is obtained while for free vibrations the boundary conditions lead to an infinite number of homogeneous linear equations. In the latter case, roots of the equation formed by equating the main determinant to zero give the frequencies of free vibrations.

As an illustration, a plate clamped in two opposite edges and with uniformly distributed load over it is considered. Particular results obtained are compared with those for bars. *S. C. Das* (Madras)

5339:

Nardini, Renato. Sui fronti d'onda nella magneto-elasticità. Rend. Sem. Mat. Univ. Padova **28** (1958), 225–243.

The author finds the wave velocity of the various magnetoelastic waves using a technique in the theory of characteristics due to Levi-Civita. The results agree with those of Baños [Phys. Rev. (2) **104** (1956), 300–305; MR **19**, 210] for plane waves. {The reviewer thinks that a third approach due to Lighthill [Philos. Trans. Roy. Soc. London. Ser. A., to be published] would also yield interesting results in this new field.}
D. R. Bland (Manchester)

5340:

Biot, M. A. Folding of a layered viscoelastic medium derived from an exact stability theory of a continuum under initial stress. Quart. Appl. Math. **17** (1959), 185–204.

This paper is a more exact treatment of a problem previously considered by the author [Proc. Roy. Soc. London. Ser. A **242** (1957), 444–454; MR **19**, 1113]. Products of the components of the initial stress and of the rotation produced by the additional stress are now included in the equilibrium equations. The case of a viscous layer embedded in a viscous medium is treated in detail. The author concludes that, in the range of significant instability, the approximate theory is applicable.
D. R. Bland (Manchester)

5341:

Lepik, Yu. R. Determination of residual deflection and residual forces in the case of unloading of elasto-plastic plates. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 154–157. (Russian)

Deformation theory of plasticity is applied to the large deflection theory of plates. Assuming that deformations and internal forces (due to the given loading q) are known, unloading is studied. For unloading von Karman's equations for flexible membranes virtually hold and variational methods are applied to calculate residual deflection under the load $q^0 < q$. An example is sketched for a case when no new plastic regions develop during unloading and the proportional loading condition is not being violated. The paper is not self-contained and numerous references are made to other papers of the same author. *A. Sawczuk* (Warsaw)

5342:

Mazing, R. I. Elasto-plastic deformation of a rapidly rotating cylinder. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 143–147. (Russian)

Piece-wise linear theory is applied to the analysis of stress and deformation of an ideal elastic-plastic rotating cylinder in plane strain. W. T. Koiter's [*Anniversary volume on applied mechanics dedicated to C. B. Biezeno*, pp. 232–251, Tech. Uitgev. H. Stam, Haarlem, 1953; MR **14**, 1148] approach to the problem of elastic-plastic equilibrium of a thick walled tube subjected to rotationally symmetric pressure is extended to the boundary conditions in question. A closed form relation between angular velocity and position of the elastic-plastic boundary is given. Bursting angular velocity is obtained for the range of validity of the yield condition $\sigma_t - \sigma_\theta = 2k$, e.g. for $0{,}162 \leq \alpha \leq 1$ (α denotes the inner/outer radii ratio). Radial displacement and axial stress are computed for the particular case of $\alpha = 0{,}5$ and compared to those obtained by numerical integration for the Prandtl-Reuss flow law. *A. Sawczuk* (Warsaw)

5343:

Druyanov, B. A. Indentation of a rigid punch in a thick plastically non-homogeneous layer. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 161–166. (Russian)

Perturbation method is applied to an approximate solution of stress and velocity characteristics in the plane problem of a flat die indentation into a plastic layer. Material is assumed to be non-homogeneous, yield point decreasing exponentially. Results can be compared with Hill's solution for a constant yield point.
A. Sawczuk (Warsaw)

5344:

Olszak, W.; und Sawczuk, A. Die Grenztragfähigkeit von zylindrischen Schalen bei verschiedenen Formen der Plastizitätsbedingung. Acta Tech. Acad. Sci. Hungar. **26** (1959), 55–57. (English, French and Russian summaries)

The paper is concerned with the load carrying capacity of orthotropic cylindrical shells under rotationally symmetric states of loading that do not involve axial membrane forces, so that the generalized stresses consist of the axial bending moment M and the circumferential membrane force N. Considering various yield conditions for the shell material, the authors determine the corresponding yield loci in the M, N-plane and suggest a method

of piecewise linearization of curved yield loci. To illustrate the application of the theory, they discuss the load carrying capacity of cylindrical water tanks of different designs. *W. Prager* (Providence, R.I.)

5345:

Ivlev, D. D. On a single particular solution of ideal plasticity theory in cylindrical coordinates. Soviet Physics. Dokl. **123** (3) (1958), 1294–1296 (988–990 Dokl. Akad. Nauk SSSR).

A family of special solutions in cylindrical coordinates is obtained for the stress and velocity fields in a homogeneous isotropic mass of rigid/plastic material with Mises yield function and flow potential. Axially symmetric body forces and additive volumetric changes (representing thermal expansion) are included. The results are left in the form of a complicated chain of implicit equations and integrals involving several disposable functions and constants. Various known solutions are contained in this family but no new solutions of explicit problems are mentioned. *R. Hill* (Nottingham)

5346:

Khuan, Ke-Chzhi [Huan, Kè-Čži]. On work-hardening of plastic solids. J. Appl. Math. Mech. **22** (1958), 758–762 (544–546 Prikl. Mat. Meh.).

W. Prager's hardening rule [J. Appl. Mech. **23** (1956), 493–496; MR **18**, 688] postulates that the yield surface of a plastic material move in a translation in the direction of the plastic strain increment. This rule is not invariant with respect to reductions in dimensions possible in almost any applications. In other words: if the yield surface in 9-space σ_{ij} moves according to Prager's rule, the two-dimensional yield locus, e.g., in plane stress σ_x, σ_y does not do so. Thus, B. Budiansky's remark [ibid. **24** (1957), 481–483] that Prager's rule is incompatible with the assumption of original isotropy, is not justified whenever the rule is correctly applied. The author points to this fact and establishes the special forms Prager's rule (as well as a combination of isotropic and Prager-hardening) takes in certain applications. Similar investigations have been carried out by R. T. Shield and the reviewer [Z. Angew. Math. Phys. **9a** (1958), 260–276; MR **20** #6859]. The results are in agreement when applied to plane stress. *H. Ziegler* (Zürich)

5347:

Zoller, K. Die Wärmeleitgleichung bei Wärmespannungen. Ing.-Arch. **28** (1959), 366–372.

Bei der Deformation des festen Körpers wird Wärme erzeugt oder verbraucht. Die Fouriersche Wärmeleitungsgleichung ist daher durch eine entsprechende Quellverteilung zu ergänzen. Dies wurde zuerst von F. Neumann durchgeführt, der ein der zeitlichen Ableitung der Volumsdilatation proportionales Glied hinzufügte. Der Verfasser untersucht nun, unter welchen Voraussetzungen die so ergänzte Gleichung vollständig ist.

Man vermißt an den recht abstrakt gehaltenen Ausführungen vor allem Näheres über die "Zustandsparameter", für die man beim Kontinuum doch wohl die Verzerrungskomponenten wählen wird. Ein Vergleich mit den Ergebnissen anderer Autoren [z.B. Biot, J. Appl. Phys. **27** (1956), 240–253; MR **17**, 1035] wird dadurch sehr erschwert. *H. Parkus* (Vienna)

5348:

Biot, M. A. New thermomechanical reciprocity relations with application to thermal stress analysis. J. Aero/Space Sci. **26** (1959), 401–408.

The author presents a generalization of Maxwell's classical reciprocity relations to the case of linear thermoelasticity. In addition to ordinary forces and displacements, temperature changes corresponding to "entropy displacements" are involved. These entropy displacements are determined by minimizing a dissipation function. The author shows how his reciprocity relations may be used to determine thermal stresses in elastic systems without evaluation of the temperature field. He illustrates the method with the problem of stationary thermal stresses in a thin circular cylinder. *W. Noll* (Pittsburgh, Pa.)

5349:

Roy, Maurice. Théorie thermodynamique condensée de l'élasticité linéaire. Ing.-Arch. **28** (1959), 277–280.

This is an exposition of standard linear thermoelasticity. *W. Noll* (Pittsburgh, Pa.)

5350:

★Parkus, Heinz. Instationäre Wärmespannungen. Springer-Verlag, Vienna, 1959. v+166 pp. $9.05.

The book constitutes a continuation of *Wärmespannungen infolge stationärer Temperaturfelder* [Springer, Vienna, 1953; MR **16**, 306] by E. Melan and H. Parkus. It generalizes the previously discussed problems to dynamical cases and some non-elastic materials. The author deals mostly with two-dimensional, axially-symmetrical, spherically-symmetrical and plate problems.

The first part of the book (Chap. I–IV) gives the basic theorems of thermoelasticity and some special solutions concerning non-steady stress distributions in elastic bodies due to the action of suddenly applied heat sources. The following body shapes are investigated: infinite and semi-infinite space, space with a spherical cavity, cylinder and cylindrical thick-walled tube, sphere and unbounded disk. The heating occurs either on the surface or in the interior of the body, heat sources being two-dimensional, one-dimensional or point-sources (a pole or dipole). All solutions obtained by means of Laplace-transform are either explicit or have the form of single series and integrals. Some of them are completed by graphs and tables of value in technical applications. Problems of variable time and moving heat sources (Chap. III, IV) have been taken into consideration, all cases being treated as quasi-static ones.

In Chap. V inertia terms in the equilibrium equations are taken into account in order to tackle the problems of thermal impact and wave effects in the bodies, previously discussed in a quasi-static manner.

Chap. VI presents a theory and some simple solutions of non-steady stresses in visco-elastic bodies of the Kelvin, Maxwell, and standard type; Chap. VII deals with elastic-plastic sphere, tube, disk and plate.

In spite of its small volume the book constitutes a complete survey (up to 1957) of all world achievements in the subject discussed and is of great value to scientists working on thermoelasticity. *M. Sokolowski* (Warsaw)

STRUCTURE OF MATTER

5351:

★Seitz, Frederick; and Turnbull, David. (Editors) Solid state physics: Advances in research and applications. Vol. 6. Academic Press Inc., New York-London, 1958. xiv + 429 pp. $12.00.

The sixth volume of *Solid State Physics Advances* contains seven articles of varied interest. M. H. Rice, R. G. McQueen and J. M. Walsh, in "Compression of Solids by Strong Shock Waves", review a branch of high-pressure physics in which P-V data are being obtained at pressures up to the order of 10^6 bars. The relevant theory of shock waves is described briefly, followed by a comprehensive review of experimental results and their analysis for an equation of state. G. Borelius, in "Changes of State of Simple Solid and Liquid Metals", is concerned largely with his important contributions to this subject, both experimentally and theoretically. He is concerned with the separation of energy, entropy and volume into vibrational and structural parts, and with showing how the discontinuous nature of melting arises and how the compressibility, activation energy for self-diffusion, etc. may be calculated. Finally, the simple situation at the melting transition for the f.c.c. metals is contrasted to that for other metals.

W. W. Piper and F. E. Williams give an extensive review of "Electroluminescence", the light emission from solids which is excited by an applied voltage. They show that many different mechanisms may arise in different cases, and in their discussion consider the overall process in three parts: excitation of the crystal, transfer of the energy through the crystal, and emission. The observed behaviour of zinc sulphide and, more briefly, some other materials is then considered from the point of view of these notions.

In "Macroscopic Symmetry and Properties of Crystals", Charles S. Smith gives an admirably concise and up-to-date account of this long-established branch of physics using tensor and matrix notation. After sketching the necessary symmetry notions, he discusses the effect of symmetry on the description of various physical properties representable by tensors of first to fourth rank. The sections on pyroelectricity, piezoelectricity and piezoresistivity are especially useful. He has also added a number of comments on the way in which the macroscopic properties may be understood in terms of the crystal structure. The notation for the physical quantities may prove rather confusing to those not familiar with the I. R. E. standards.

In reviewing "Secondary Electron Emission", A. J. Dekker has stressed the experimental information available concerning the emission of electrons from surfaces bombarded with electrons. However, he has also given a brief discussion of the several theories in this field.

Finally there are two articles on optical properties. In the first, on "Optical Properties of Metals", M. P. Givens considers briefly the classical accounts and some of the contributions from quantum mechanics. This is followed by a review of the experimental work in the field. The other article, entitled "Theory of the Optical Properties of Imperfections in Nonmetals", is by D. L. Dexter. This is mainly a theoretical paper in which it is shown how the general theory of the interaction of radiation (treated classically) with matter (treated quantum-mechanically) can be applied, using various approximations and simplifications, to the theory of color centres in non-metals. It is supplemented by a few sections on diverse topics such as anisotropic effects, the optical properties of dislocations and colloid particles, etc.

M. S. Patterson (Berkeley, Calif.)

5352:

★Seitz, Frederick; and Turnbull, David. (Editors) Solid state physics: Advances in research and applications. Vol. 9. Academic Press, New York-London, 1959. xv + 548 pp. $14.50.

The ninth volume of this Advances series begins with an article on "The electronic spectra of aromatic molecular crystals" by H. C. Wolf. He reviews the measured absorption and fluorescence spectra (illustrated largely from his own work) for a considerable number of such crystals, as well as mentioning many details of experimental technique. The theoretical situation is discussed in general terms on the basis of a model of weak coupling between molecules. "Polar semiconductors" by W. W. Scanlon is concerned with lead sulphide, lead selenide and lead telluride, compounds having some of the ionic character of alkali halides and some of the homopolar character of germanium and silicon. The paper reviews what is known of their general, physico-chemical, optical, and electrical properties. D. J. Montgomery has undertaken the difficult task of presenting the present state of knowledge on the ancient subject of "Static electrification of solids" and showing the nature of the experimental difficulties. He also puts forth some theoretical ideas that might influence future lines of experiment.

The borderlands of nuclear and solid state physics are explored by E. Heer and T. B. Novey in "The interdependence of solid state physics and angular distribution of nuclear radiations". They discuss the various factors in a solid or liquid (electric quadrupole interactions, internal magnetic fields, paramagnetism, etc.) which may influence the angular correlations of successive radiations from a decaying nucleus and point out that such studies may in turn throw light on solid state problems.

A. H. Kahn and H. P. R. Frederikse review the "Oscillatory behaviour of magnetic susceptibility and electronic conductivity" as a function of magnetic field, setting out the theory of the de Haas-van Alphen effect and of various transport properties, especially electrical conductivity, in the presence of a magnetic field. Also included is a brief resumé of the theory of the electronic levels in a periodic lattice in a magnetic field. Finally the experimental measurements on bismuth and some other substances are reviewed.

The remaining two papers take up the second half of this volume. "Heterogeneities in solid solutions" by A. Guinier is concerned with the departures from random distribution of solute atoms in metallic alloys which have been the object of much x-ray diffraction work. The principles of the x-ray methods are first discussed. The greater part of the article is devoted to the situation in supersaturated solid solutions before resolvable precipitate particles appear. Some of the physical properties of such systems are also discussed.

The first half of "Electronic spectra of molecules and ions in crystals: Part II. Spectra of ions in crystals" by D. S. McClure is theoretical, and treats spectra arising in situations where a tight binding approximation may be

used: firstly, spectra arising from inner electrons in the presence of the crystal field and, secondly, spectra associated with "charge transfer" in groups of ions for which molecular orbital theory is introduced. The experimental findings are then discussed in relation to this background. *M. S. Patterson* (Berkeley, Calif.)

5353:

Fischer, K. Die Temperaturabhängigkeit der Struktur punktförmiger Fehlstellen in kubischen Kristallgittern. Z. Physik **157** (1959), 198–218.

"Die Struktur einer Gitterfehlstelle ist bei gegebener Temperatur and verschwindenden äußeren Kräften durch die Forderung bestimmt, daß die freie Energie des gestörten Kristalls in Abhängigkeit von den mittleren Atomlagen minimal ist. Sie läßt sich daher durch ein Variationsverfahren ermitteln. Die freie Energie kann man hierbei nach einem Näherungsverfahren berechnen, das schon zur Ermittlung der thermischen Eigenschaften von idealen Kristallen mit Erfolg verwendet wurde."

Aus der Zusammenfassung des Autors

5354:

Kirzhnits, D. A. The limits of applicability of the quasi-classical equation of state of matter. Soviet Physics. JETP **35** (8) (1959), 1081–1089 (1545–1557 Ž. Eksper. Teoret. Fiz.).

The calculation of the equation of state of matter by use of the statistical (Thomas-Fermi) theory implies the neglect of certain characteristic quantum effects depending on the wave nature of the electron, as well as of exchange effects arising from the electron spin. The author examines the orders of magnitude of the corrections needed in the calculation of the compressibility, making use of the expression for the strain tensor in terms of the electron density function. Estimates are obtained for the corrections in various regions of temperature and electron density. *E. L. Hill* (Minneapolis, Minn.)

5355:

Carruthers, Peter. Scattering of phonons by elastic strain fields and the thermal resistance of dislocations. Phys. Rev. (2) **114** (1959), 995–1001.

A theory of phonon scattering by elastic strain distributions is presented. Their Fourier components play a role similar to that of the potential in external field approximation. The strain is treated on continuum basis; otherwise, only atomic characteristics enter, allowing examination of the effects of crystal structure, interatomic potentials, etc., and scattering between different polarization modes. A Boltzmann equation is found. The results are applied to estimation of the low temperature resistance in insulators due to dislocations, which devolves upon a characteristic relaxation time τ. For an edge dislocation an approximation gives $\tau^{-1} \propto \sigma q [\ln (nb^{-1}\sigma^{-1/2})]^2$ with σ-dislocation density, q-phonon wave vector, n-average number in slip plane, b-Burger vector, whereas the log-term is missing for screw dislocations. The associated order-of-magnitude difference in scattering effect between the two types is borne out experimentally. *H. G. Baerwald* (Albuquerque, N.M.)

5356:

Filipovich, V. N. Theory of x-ray scattering by distorted crystals. I. Theory without atomic factors. Soviet Physics. Tech. Phys. **28** (3) (1958), 2486–2495 (2716–2726 Ž. Tehn. Fiz.).

By expanding the electron density into a Fourier series the author obtains formulas for the amplitudes and intensities of x-rays scattered by distorted crystals, i.e., crystals containing internal cavities, cracks, and deformations. Several types of distribution functions $H(r, t)$, used to describe the nature of the distortion, illustrate the method and results.

H. A. Hauptman (Washington, D.C.)

5357:

Filipovich, V. N. Theory of x-ray scattering by distorted crystals. II. Theory with atomic factors. Soviet Physics. Tech. Phys. **28** (3) (1958), 2496–2506 (2727–2738 Ž. Tehn. Fiz.).

Certain approximations and other difficulties inherent in the theory described in the previous paper are avoided by the introduction of the known atomic scattering factors of the atoms constituting the (distorted) crystal. This amounts to the assumption that the electron density in a crystal can be represented as a sum of the electron densities of the constituent atoms. The general theory of the previous paper now yields more accurate, but somewhat more complicated, expressions for the amplitudes and intensities of the scattered radiation.

H. A. Hauptman (Washington, D.C.)

FLUID MECHANICS, ACOUSTICS

See also 5148, 5475, 5489.

5358:

Moreau, Jean-Jacques. Invariants intégraux d'un ensemble de solutions des équations de l'hydrodynamique. C. R. Acad. Sci. Paris **248** (1959), 771–773.

The author considers an n-parameter family of isochoric circulation-preserving motions. Restricting this family so that each member has the same circulation, at each instant, about any given spatial circuit, he finds an exterior differential form over the family which is invariant in time. *C. Truesdell* (Bloomington, Ind.)

5359:

Takano, Kenzo. Effets de second ordre d'un seuil semi-indéfini sur une houle irrotationnelle. C. R. Acad. Sci. Paris **249** (1959), 622–624.

5360:

Tasca, Dan. Remarques sur les relations entre les tensions et les vitesses de deformations unitaires dans le mouvement laminaire des fluides réels incompressibles. Bul. Inst. Politehn. Bucureşti **20** (1958), no. 2, 61–64. (Russian, English and German summaries)

5361:

Yakimov, Yu. L. Unsteady motions of incompressible fluid in narrow regions. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 1080–1083. (Russian)

5362:

Power, G. Accelerating body with vortex trail in variable two-dimensional flow. Z. Angew. Math. Mech. **39** (1959), 139–146. (German, French and Russian summaries)

Using a method developed by R. M. Morris [Proc. Roy. Soc. London. Ser. A. **164** (1938), 346–368; **188** (1947), 439–463; MR **8**, 542] the author finds expressions for the resultant hydrodynamical forces on a cylinder of general cross-section when it is in motion in a fluid which is itself moving non-uniformly, and when a vortex trail is assumed to form behind the cylinder. The undisturbed motion of the fluid is attributed to a number of varying spiral vortices, which can be used as an approximation to any given flow. *R. M. Morris* (Cardiff)

5363:

Zarea, Stefan. Le mouvement hélicoïdal Gromeca-Beltrami d'un fluide parfait dans une conduite cylindrique circulaire. Bul. Inst. Politehn. București **20** (1958), no. 2, 65–69. (Russian, English and German summaries)

5364:

Clarke, Joseph H. On the application of the Ursell-Ward theorem to wings with edge forces. J. Aero/Space Sci. **26** (1959), 535–536.

The edge forces treated here are of two kinds, both due to square-root singularities in the velocity obtained from linearized theory. The first kind occurs on subsonic edges not satisfying the Kutta condition, the second kind on edges which exhibit parabolic bluntness. The author considers various effects of edge forces on the relation between the drag in a given flow and the drag in the reversed flow. *F. Ursell* (Cambridge, England)

5365:

Karas, Karl. Stationäre Laminarströmungen durch Kanäle von elliptischem Querschnitt bei konstantem Druck oder statischer Druckverteilung. Ing.-Arch. **28** (1959), 117–153.

5366:

van Dantzig, D. Einige analytische Ergebnisse über die Wasserbewegung in einem untiefen Meere. Z. Angew. Math. Mech. **39** (1959), 169–179.

A survey of mathematical work undertaken since 1953 by members of the Mathematisch Centrum, Amsterdam, on the motion in a shallow basin under the influence of applied wind forces, bottom friction, and the rotation of the earth. *F. Ursell* (Cambridge, England)

5367:

Karas, Karl. Stationäre Laminarströmungen durch Kanäle von Rechtecksquerschnitt bei konstantem Druck oder statischer Druckverteilung. Z. Angew. Math. Mech. **39** (1959), 146–160. (English, French and Russian summaries)

5368:

Takano, Kenzo. Un exemple de calcul d'une seiche portuaire d'interaction de second ordre. C. R. Acad. Sci. Paris **249** (1959), 222–223.

5369:

Takano, Kenzo. Sur les oscillations du deuxième ordre des liquides dans les bassins portuaires à profondeur constante. C. R. Acad. Sci. Paris **249** (1959), 30–32.

5370:

Cherry, T. M. Some nozzle flows found by the hodograph method. J. Austral. Math. Soc. **1** (1959/61), part 1, 80–94.

The author demonstrates that by superposition of his earlier solution [Philos. Trans. Roy. Soc. London Ser. A **245** (1953), 583–626; MR **14**, 921] with other simple solutions, a variety of nozzle flows in the neighborhood of the throat can be obtained. *Y. H. Kuo* (Peking)

5371:

Aslanov, S. K.; and Legkova, V. A. Outflow of a gas jet from a vessel of finite size. J. Appl. Math. Mech. **23** (1959), 266–272 (190–193 Prikl. Mat. Meh.).

5372:

Ladyzhenskaya, O. A. Stationary motion of viscous incompressible fluids in pipes. Soviet Physics. Dokl. **124** (4) (1959), 68–70 (551–553 Dokl. Akad. Nauk SSSR).

The author shows that for an infinite pipe of arbitrary cross-section, the ends of which are cylindrical, at least one laminar motion of viscous incompressible fluid exists for any Reynolds number. The proof uses a technique of functional analysis, the required solution existing because a solution of a generalized equation, which reduces to the Navier-Stokes equation under the conditions of the problem, exists. *W. D. Collins* (Newcastle-upon-Tyne)

5373:

Kawaguti, Mitutosi. Note on Allen and Southwell's paper "Relaxation methods applied to determine the motion, in two dimensions, of a viscous fluid past a fixed cylinder". Quart. J. Mech. Appl. Math. **12** (1959), 261–263.

In this note, the results of Allen and Southwell [same J. **8** (1955), 129–145; MR **16**, 1171] are compared with hitherto known results and shown to be numerically incorrect. *Y. H. Kuo* (Peking)

5374:

Christopherson, D. G.; and Dowson, D. An example of minimum energy dissipation in viscous flow. Proc. Roy. Soc. London, Ser. A **251** (1959), 550–564.

This very interesting paper is concerned with the slow motion of a sphere down a vertical tube, which has a diameter slightly greater than that of the sphere and is filled with viscous fluid. The forces on the sphere are calculated on the basis of lubrication theory when it is both eccentrically placed in the tube and rotating. The authors then

show that the maximum rate of fall of the sphere occurs when it is in an eccentric position and has a determinate angular velocity. This maximum rate of fall, which corresponds to minimum energy dissipation, is about twice the rate of fall corresponding to zero eccentricity.

Experiments to test the theory are described. The agreement in one case is excellent and in others fairly good; reasons for discrepancies are put forward and discussed. *K. Stewartson* (Durham)

5375:

Nevzglyadov, V. G. A contribution to the problems of flow past solid bodies on the basis of the phenomenological theory of turbulence. Vestnik Leningrad. Univ. **13** (1958), no. 19, 156–169. (Russian. English summary)

La dynamique d'un fluide visqueux ne présente pas une théorie aussi complète que celle d'un fluide idéal.

L'auteur essaie de faire une étude générale en se basant sur sa théorie phénoménologique de la turbulence [voir l'article, même Vestnik **3** (1948), no. 3, 3].

Avant tout il s'agit de trouver la force R moyenne qui agit sur le corps immobile par le courant extérieur. On trouve cette force en appliquant la loi de variation de la quantité du mouvement. Ensuite l'auteur linéarise les équations du mouvement extérieur avec les conditions aux limites à l'infini.

En utilisant le premier principe de la thermodynamique on trouve une généralisation du vecteur de Umov dans un mouvement turbulent.

Finalement l'auteur montre qu'on peut présenter les équations linéarisées du mouvement dans le courant extérieur, pour une turbulence extérieure constante, sous la forme d'un problème des variations des équations Euler-Lagrange. *M. Kiveliovitch* (Paris)

5376:

★Batchelor, G. K. (Editor) The scientific papers of Sir Geoffrey Ingram Taylor. Vol. II. Meteorology, oceanography and turbulent flow. Cambridge University Press, New York, 1960. x+515 pp. (3 plates) **$14.50.**

A collection of 44 papers, dating from 1915 to 1956. Volume I, on mechanics of solids (1958), was listed in MR **19**, 826.

5377:

Serrin, James. On the uniqueness of compressible fluid motions. Arch. Rational Mech. Anal. **3**, 271–288 (1959).

Cet article concerne l'unicité de la solution du problème aux valeurs initiales pour le mouvement, dans un domaine borné $\mathscr{V}(t)$, d'un fluide compressible. Il est montré qu'avec un choix convenable des données à la frontière, à tout instant, le mouvement est déterminé d'une manière unique, par la connaissance de la répartition initiale de la vitesse, de la température et de la densité. Ce travail généralise les résultats de Graffi [Rev. Un. Mat. Argentina **17** (1956), 73–77; MR **18**, 617] en ce sens qu'aucune hypothèse n'est faite concernant une forme particulière de l'équation d'état. Les démonstrations procèdent des méthodes développées par Hadamard, Zaremba, Friedrichs et Lewy, Orr, Foa, avec une utilisation originale de l'équation de transfert. Sous la forme qui leur est donnée, les résultats ne s'appliquent pas aux mouvements avec onde de choc, mais il pourraient être étendus sans difficulté à de tels mouvements. Les deux théorèmes d'unicité centraux concernent, l'un les fluides compressibles non visqueux, l'autre les fluides compressibles, visqueux, conducteurs de la chaleur avec λ, μ, κ constants et $\mu>0$, $\kappa\geqq 0$, $3\lambda+2\mu>0$. Les cas d'exception $3\lambda+2\mu=0$ et $\lambda=\mu=0$, $\kappa>0$ sont aussi traités. Enfin, le cas où μ est une fonction de la température, est discuté. *R. Gerber* (Grenoble)

5378:

Bader, W. Zur ebenen Strömung. Z. Angew. Math. Mech. **39** (1959), 163–164.

The steady irrotational subsonic plane motion of a non-viscous gas past a body is considered when the velocity at infinity has the same value in all directions. The equation is transformed into orthogonal curvilinear coordinates ξ and η so that the boundary of the body is given by $\eta=0$. The stream function ψ is so chosen that the values of ψ_ξ, ψ_η vanish at infinity for $\xi=\pm\infty$ and or $\eta=\pm\infty$. The corresponding differential equation for ψ becomes $\nabla\psi-a\psi_\xi-b\psi_\eta=b$, $a(\xi,\eta)=\rho_\xi/\rho$, $b(\xi,\eta)=\rho_\eta/\rho$. This is solved by an iterative procedure combined with the Ritz-Galerkin method.

The author is apparently unaware that this problem can be reduced to the solution of the differential equation

$$\nabla^2\psi_1/\rho_1 = \nabla^4\rho_1/\rho_1, \quad \psi_1=\rho_1\psi_1, \quad \rho_1=\rho^{-1/2}$$

[B. R. Seth, Bull. Calcutta Math. Soc. **46** (1954), 217–230; MR **17**, 312]. *B. R. Seth* (Kharagpur)

5379:

Ting, Lu. On the integral of moment of pressure in supersonic flow. J. Aero/Space Sci. **26** (1959), 758–759.

5380:

Wan, Koon-Sang. On an exact solution of incompressible, inviscid, hypersonic, stagnation-point flow for a sphere. J. Aero/Space Sci. **26** (1959), 755–756.

5381:

Gubkin, K. E. Propagation of discontinuities in sound waves. J. Appl. Math. Mech. **22** (1958), 787–793 (561–564 Prikl. Mat. Meh.).

Consider a weak disturbance advancing into a steady compressible flow. By integrating along particle paths or along orthogonal trajectories of sound waves and neglecting terms of second order in velocity, pressure, and density perturbations, the author finds approximate solutions of the characteristic equations for unsteady flow. In particular this yields, except for an undetermined factor α, the pressure perturbation Δ on any forward moving sound wave N_+ which once coincided with the weak shock N. From the characteristic equations and a knowledge of Δ one can calculate the length $l(t)$ of a ray from N to N_+ to within an additive term $f(\alpha)$, determined by the initial conditions for the wave N under consideration. If one considers the weak shock N to be approximately a sound wave, then from the relation between Δ at the shock and shock velocity one can determine dl/dt approximately. Substitute herein the previously found $l(t)$ to obtain for $\alpha(t)$ a differential equation that can be

solved by quadratures. This determines the variation of the pressure profile of N. *J. H. Giese* (Aberdeen, Md.)

5382:

Gundersen, Roy. A perturbation analysis of shock flow in a nozzle. J. Aero/Space Sci. **26** (1959), 763.

5383:

Lyubimov, G. A. The influence of viscosity and heat conductivity on the flow of a gas behind a strongly curved shock wave. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1958**, no. 5, 33–35. (Russian)

An extension of the work (1) of Sedov, Mihaĭlova, and Černyĭ [Vestnik Moscov. Univ. Ser. Fiz.-Mat. Estest. Nauk 8 (1953), no. 3, 95–100; MR **15**, 479] which dealt with the same problem in the plane and axisymmetric cases. Both papers are based on modified shock relations that state the equality ahead and behind of a (stationary) shock of the expressions $\mathbf{p}_n - \rho v_n \mathbf{v}$ (momentum) and $\mathbf{p}_n \cdot \mathbf{v} + k\partial T/\partial n - \rho(v_n(v^2/2+e)$ (energy). Here $\mathbf{p}_n$ is the viscous-stress vector acting on the element with surface normal $\mathbf{n}$. Uniform flow being assumed upstream ($\mathbf{v}_1$), the derivatives of $\mathbf{v}_2$ appearing in the shock relations can be expressed approximately in terms of v_{n1}/v_{n2} and of the principal radii of curvature $R_{\alpha,\beta}$ of the shock. Correction terms for the ratios of the stagnation temperatures and stagnation pressures are obtained; they include the factor $(1/Re_\alpha + 1/Re_\beta)$, where $Re_{\alpha,\beta}$ are Reynolds numbers formed with the downstream flow quantities and $R_{\alpha,\beta}$. In addition, the formulas obtained for those ratios in reference (1) for the axisymmetric case are presented in revised form.

{Reviewer's note: The modified shock relations appear in Sedov's book *Ploskie zadači gidrodinamiki i aèrodinamiki* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1950; MR **19**, 376; Ch. VIII]. They are obtained by applying momentum- and energy-conservation laws to an established discontinuity surface in a viscous fluid, a procedure which seems open to objection on the grounds of accepted shock-structure theory. Moreover, recent calculations by Probstein and Kemp [Avco Res. Rep. no. 48 (1959)] on the basis of Navier-Stokes shock-structure indicate that not all correction terms of order $1/Re$ can be obtained in this way.}

G. Kuerti (Cleveland, Ohio)

5384:

Kolotikhina, Z. V. On the vibrations of a cylindrical shell in water and the complex acoustical spectrum of its radiation. Soviet Physics. Acoust. **4** (4) (1958), 344–351 (333–340 Akust. Ž.).

The author considers the interaction between an ideal compressible fluid medium and the axially symmetric vibrations of a cylindrical elastic shell under an arbitrary external periodic load. The usual acoustic approximations and linearizations are made and the free and forced axially symmetric vibrations are obtained through separation of variables in cylindrical space coordinates and time. The velocity potential in the fluid medium is obtained in terms of the natural frequencies of the shell in the fluid medium. These frequencies are obtained, in turn, through the natural frequencies of the shell in vacuum. No specific example is presented.

P. Chiarulli (Chicago, Ill.)

5385:

Polyanskaya, V. A. The field of a pulse radiator in an underwater sound channel. Soviet Physics. Acoust. **5** (5) (1959), 91–99 (91–100 Akust. Ž.).

The field mentioned is expressed as a sum of normal waves and is calculated by the WKB method for various pulse shapes. The effects of the pulse shape and duration on the size of the region of cylindrical attenuation (i.e., as $1/r$) are examined. *R. N. Goss* (San Diego, Calif.)

5386:

Kubanskii, P. N. The effect of acoustic streaming on convective heat exchange. Soviet Physics. Acoust. **5** (5) (1959), 49–55 (51–57 Akust. Ž.).

Previous results on heat transfer from heated oscillating cylinders are explained by effects due to secondary streaming flow. New similarity parameters for this effect are given for both free and forced convection and are claimed to collapse data from experiments on heated cylinders in strong acoustic fields.

H. C. Levey (Nedlands)

5387:

Talwar, S. P. Hydromagnetic stability of a conducting, inviscid, incompressible fluid of variable density. Z. Astrophys. **47** (1959), 161–168.

The author investigates the Rayleigh instability of an inviscid, incompressible, perfectly conducting fluid of variable density in the presence of a horizontal magnetic field. The magnetic field is found to have a stabilizing influence and it is shown that there are now stable configurations which would be unstable in the absence of a field. *H. Greenspan* (Cambridge, Mass.)

5388:

Chao, Kai-Hua. Surface oscillations of a charged column in a longitudinal magnetic field. Soviet Physics. JETP **35** (8) (1959), 1031–1034 (1475–1480 Ž. Eksper. Teoret. Fiz.).

This paper presents an analysis of the surface oscillations and stability of a charged hydrodynamic column (pinch) in an external field. The surface-wave dispersion formula is derived and features of the oscillation spectrum are investigated. *H. Greenspan* (Cambridge, Mass.)

5389:

McCune, J. E.; and Sears, W. R. On the concepts of moving electric and magnetic fields in magnetohydrodynamics. J. Aero/Space Sci. **26** (1959), 674–675.

In this short note, the authors discuss certain misleading statements in the current literature concerning moving uniform magnetic fields. They wish to emphasize that, "In magnetohydrodynamics, . . . when we speak of the induced electromotive force $\mathbf{q}\times\mathbf{B}$, we must keep clearly in mind that this effective field does not arise from the motion of the conductor 'relative to the magnetic field', but, rather it arises from the motion (at velocity $\mathbf{q}$) of the conducting fluid particle relative to the system in which the observer measures the electric field." Although this point has been extensively discussed in many physics texts it is perhaps worthwhile to bring it again to the attention of fluid dynamicists.

H. Greenspan (Cambridge, Mass.)

5390:

Carstoiu, John. Sur le mouvement lent d'un fluide visqueux conducteur entre deux plans parallèles. C. R. Acad. Sci. Paris **249** (1959), 1192–1195.

5391:

Marra, Teresa. Il teorema di reciprocità per l'equazione della magneto-idrodinamica unidimensionale. Atti Accad. Ligure **14** (1958), 188–193. (English summary)

On obtient au moyen d'une transformation de Laplace un théorème de réciprocité pour les solutions de l'équation

$$\frac{\partial^2 y(z,t)}{\partial t^2} = 2F\frac{\partial^3 y(z,t)}{\partial t\,\partial z^2} + 5^2\frac{\partial^2 y(z,t)}{\partial z^2}$$

que l'on rencontre en magnétohydrodynamique et dans la théorie des oscillations longitudinales d'une corde avec friction interne. Dans ce dernier cas on a en particulier le résultat suivant: les extrémités étant fixées si l'on considère deux systèmes (1) et (2) correspondant à des écarts initiaux nuls, et à une vitesse initiale non nulle dans un petit intervalle autour des points a et b respectivement, à l'instant t le déplacement en a du système (2) est égal au déplacement en b du système (1). *J. Naze* (Marseille)

5392:

Carini, Giovanni. Il teorema dell'energia nella magneto-idrodinamica dei fluidi viscosi comprimibili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 470–473.

Le théorème de l'énergie pour un fluide visqueux compressible, conducteur d'électricité, se déplaçant dans un champ magnétique est obtenu à partir du système des équations de Maxwell, de la loi d'Ohm, et des équations de Navier-Stokes. On met en évidence la pression $-\mu/8\pi H^2$ due à la force de Lorentz, ainsi que le terme d'effet Joule. Pour que la dissipation d'énergie soit nulle il faut en particulier que le fluide soit conducteur parfait.

J. Naze (Marseille)

5393:

Agostinelli, Cataldo. Sull'equilibrio relativo magneto idrodinamico di masse fluide elettricamente conduttrici uniformemente rotanti e gravitanti. Boll. Un. Mat. Ital. (3) **14** (1959), 95–101.

Cet article est consacré à la recherche des conditions que doit vérifier le champ magnétique pour qu'une masse de fluide incompressible, doué d'une conductivité électrique infinie, placée dans un champ gravitationnel et en rotation uniforme autour d'un axe admette pour figure d'équilibre relatif un ellipsoïd de révolution. On trouve en particulier que le champ magnétique doit être symétrique autour de l'axe de rotation. Diverses conditions sont obtenues suivant que l'on suppose le champ magnétique méridien ou non, les solutions correspondantes devant être développées dans un article ultérieur.

J. Naze (Marseille)

5394:

Surin, Aline. Sur le tenseur d'impulsion-énergie dans le schéma fluide parfait en théorie de Jordan-Thiry. C. R. Acad. Sci. Paris **247** (1958), 2304–2306.

"Nous voulons définir le schéma fluide parfait de la théorie de Jordan-Thiry susceptible de généraliser formellement le schéma fluide parfait de la relativité générale. Pour cela, nous introduirons dans le tenseur d'impulsion-énergie, outre le terme $rV_\alpha V_\beta$, un terme de pression."

Résumé de l'auteur

5395:

Surin, Aline. Conditions de conservation dans le schéma fluide parfait de la théorie de Jordan-Thiry. C. R. Acad. Sci. Paris **248** (1959), 1476–1478.

Scopo della Nota [continuazione del #5394] è di mostrare come, nella teoria relativistica dei fluidi perfetti dovuta a Jordan e Thiry, si esprimono nella varietà V_4 le condizioni di conservazione ottenute inizialmente nella varietà V_5. *G. Lampariello* (Rome)

5396:

Ptitsyn, O. B. Hydrodynamics of polymer solutions. I. Diffusion and sedimentation of branched molecules. Soviet Physics. Tech. Phys. **29** (**4**) (1959), 65–81 (75–93 Ž. Tehn. Fiz.).

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 5073, 5074, 5075, 5304, 5307, 5339, 5389, 5476, 5501.

5397:

Jurek, Bohumil. A method for the numerical solution of ordinary differential equations of the first order and its application to geometrical optics. Apl. Mat. **4** (1959), 203–210. (Czech. Russian and French summaries)

A method is described which makes it possible to obtain an improvement in approximate solutions of first order ordinary differential equations. The method is applied to the design of aspheric surfaces.

E. Wolf (Rochester, N.Y.)

5398:

Pekar, S. I. On the theory of absorption and dispersion of light in crystals. Soviet Physics. JETP **36** (**9**) (1959), 314–323 (451–464 Ž. Eksper. Teoret. Fiz.).

Als Fortsetzung und Erweiterung einer früheren theoretischen Arbeit [S. I. Pekar, Soviet Physics. JETP **6** (1958), 785–796; MR **20** #638] wird die Lebensdauer der Exzitonen jetzt nicht mehr als unendlich lang angenommen, was die Berechnung der Absorptionserscheinungen ermöglicht. Die Exzitonenzustände werden mit Hilfe der Quantenzahl m charakterisiert, da weiter nach der neuen Annahme deren Energie sich in thermische Energie des Gitters verwandeln kann, so wird für diese Zustände die Quantenzahl q eingeführt. Ein Übergang findet nur dann statt, wenn die Energien dieser Zustände ganz verschiedener Art voneinander kaum verschieden sein werden. Der Verfasser nennt die von der Lichtwelle unmittelbar verursachte Polarisation synchrone Polarisation und die von der in Wärmewellen umgewandelten Energie verursachte thermische Polarisation. Das sind bekannte Sachen.

Auf dieses Problem werden dann die bekannten Formeln der quantenmechanischen Störungstheorie angewandt, wobei das Problem der Berechnung des Integrals

$$(1)\quad H^{a}_{mm'} = \sum_p \int H_{mq}H_{qm'}\,\frac{1-\exp\{-i(E_q/\hbar-\omega)t\}}{\hbar\omega - E_q}\,\rho_p(E_q)\,dE_q$$

entsteht. H bedeutet in (1) den Hamiltonschen Operator und $\rho_p(E_q)$ die Dichte der Zustände. Alle anderen Symbole haben die gewohnte Bedeutung. Mit Hilfe von unwesentlichen Vereinfachungen und einer komplexen Integration folgt

$$(2) \qquad H^{a}_{mm'} = \sum_{p} \left\{ P \int \frac{H_{mq}H_{qm'}}{\hbar\omega - E_q} \rho_p(E_q) dE_q - i\pi\rho_p(\hbar\omega)[H_{mq}H_{qm'}]_{E_q=\hbar\omega} \right\},$$

wo P den Hauptwert des Integrals bedeutet. (2) geht in den Resonanznenner ein, der demzufolge statt der klassischen Formel $E_m - \hbar\omega$ jetzt die Form $E_m + H_{mm}{}^a - \hbar\omega$ annimmt. (In der ganzen Arbeit wird $E_0 = 0$ gesetzt.) Da nach (2) $H_{mm}{}^a$ komplex ist, so wird der Brechungsindex ebenfalls komplex, was die Berechnung der Absorptionserscheinungen ermöglicht. Sonst bleiben die in der zitierten Arbeit hergeleiteten Resultate erhalten.

Zuletzt wird noch die Theorie der longitudinalen Wellen weiter ausgearbeitet, für die folgt $\mathbf{E}' = -4\pi\mathbf{P}$ und $\mathbf{D} = \mathbf{H} = \mathbf{E}_\perp = \mathbf{A} = 0$. Also sind dies keine elektromagnetischen Wellen mehr. *T. Neugebauer* (Budapest)

5399:

Beran, Mark J. Determination of the intensity distribution resulting from the random illumination of a plane finite surface. Opt. Acta **5** (1958), 88–92. (French and German summaries)

The propagation of a two point correlation function of a wave field is investigated. The results are used to determine the intensity distribution resulting from random illumination of a plane finite surface. The distribution at some distance away from the surface is found to differ in general from the Lambertian distribution.

E. Wolf (Rochester, N.Y.)

5400:

Wilhelmsson, Hans. On the properties of the electron beam in the presence of an axial magnetic field of arbitrary strength. Chalmers Tekn. Högsk. Handl. no. 205, 32 pp. (1958).

This paper contains a mathematical analysis of the effect of an axial magnetic field on the propagation of electromagnetic waves in an electron beam subject to the following conditions: (a) the direct current density (dc) and the dc-component of the electron density remain constant over the cross section area of the beam and uniform in the axial direction, (b) the electron beam is cylindrical in form and of infinite length. A further assumption is that the alternate current (ac) components are small in comparison to the dc-components and electron collisions are neglected.

On the basis of these assumptions, the author has investigated the solution of a pair of coupled partial differential equations which are derived from Maxwell's equations, for three cases; (I) when the Lorentz drift and relativistic terms are neglected, (II) the former terms are included, but not the latter and (III) when both the Lorentz drift and relativistic terms are included in the solution of the equations. The mathematical formulas which have been derived are too long to indicate here. However, the following results and interpretations can be drawn from this analysis. First, the Lorentz drift terms and the relativistic terms enter in different forms in the formulas and thus play different roles in the mode of propagation of the waves in the cylindrical tube, even if the order of their magnitude should be the same. For moderate velocities of the electrons in the beam, the Lorentz drift predominates. On the other hand, at high velocities, the relativistic terms have to be considered as well in the calculations of the propagation coefficients. It is shown that for large cross sections of the beam, the determination of the propagation coefficients, which correspond to various types of excitation waves, can be carried out without much difficulty. They contain certain factors which correspond to magneto-ionic waves perturbed by the drift velocity of the beam and others which represent longitudinal space charge waves (lscw). The expressions for the propagation coefficients calculated for case I are different from those of case II and III, even when the terms containing the square of the ratio of the drift velocity of the electrons in the beam to that of free space velocity (light), v_0^2/c_0^2 are neglected in the formulas of case II and III. On the other hand the number of roots (γ) determining the type of waves (propagating) remain the same in all three cases, if drift terms are left out from these formulas, a fact of considerable importance in applying these to specific practical cases. For cyclotron frequency, $\omega_H = 0$, the (lscw) disappear in case I and also in case II and III if v_0^2/c_0^2 is neglected in the formulas corresponding to these cases. However, the (lscw) are always excited even when $\omega_H = 0$, for a beam traveling through an ionized medium, on account of the combined effect of the drift terms and the relativistic corrections. They disappear in case III, when the medium is not ionized, but not in case II. This implies that under certain conditions both relativistic and drift terms must be included in the analysis of various modes of propagation of waves in such systems. This is followed by a brief discussion of the limiting cases $\omega_H = 0$ and $\omega_H = \infty$ and the type of excitations produced in the beam. When the beam of uniform density is limited within a circular wave guide with perfectly conducting wall, the boundary conditions lead to a rather complicated equation for the calculation of various modes of propagation waves within the guide (determination of the characteristic values) in the presence of an axial magnetic field. This equation takes account not only of the effect of the ionized medium, but also relativistic corrections and angular asymmetric excitation. For obvious reasons the study of the characteristic values is not undertaken. Furthermore, the author has found results obtained by others to be special cases of the formulas derived in this paper. A more detailed discussion of space charge waves in the presence of an axial magnetic field of arbitrary strength for moderate velocities of the electron and the corresponding types of mode propagation is included. Finally, the two appendices contain explicit expressions of the field components for the three cases treated in the text.

N. Chako (Flushing, N.Y.)

5401:

Bonshtedt, M. E. Calculation of electromagnetic field of system of diaphragms. Soviet Physics. Tech. Phys. **28** (**3**) (1958), 1660–1667 (1801–1808 Ž. Tehn. Fiz.).

[The Russian original was listed as MR **20** #2171.] In this article the author has calculated the electric field distribution for a system of plane parallel screens (maintained at constant potential) with coaxial apertures

placed at distances comparable to the diameters of the apertures. The potential between two consecutive screens is expressed in the form

$$(1)\quad \varphi_{n,n+1} = V_n+(V_{n+1}-V_n)\left(\frac{z-z_n}{z_{n+1}-z_n}\right)+W_n^2+W_n^1,$$

where V_n, V_{n+1}, z_n, z_{n+1} denote the potential and axial coordinates of screens and $W_n^{1,2}$ are functions which satisfy Laplace's equation and the boundary conditions

$$(2)\quad \begin{aligned} W_n^1(r, z_n) &= f_n(r), \quad r < R_n,\\ &= 0, \quad r > R_n \quad (W_n^1(r, z_{n-1}) = 0),\\ W_n^2(r, z_n) &= f_n(r), \quad r < R_n,\\ &= 0, \quad r > R_n \quad (W_n^2(r, z_{n+1}) = 0).\end{aligned}$$

Expressing $W_n^{1,2}$ in integral representation form which satisfy (2), an integral equation for the distribution of the potential function, $f_n(r)$, in the aperture is derived from the condition of continuity of the field component across the aperture of radius R_n, of the nth screen. When the aperture distribution is expressed in the form

$$(3)\quad f_n(r) = \sqrt{(R_n^2-r^2)}\sum_{k=0}^{\infty} a_{kn}\left(\frac{r}{R_n}\right)^{2k},$$

the integral equation reduces to an infinite system of equation in the unknowns a_{kn}. By limiting the system to a single aperture between two screens, one placed at a large distance in comparison to the radius of the aperture (immersion lens arrangement), the potential is calculated up to third order terms in the series (3) when the cathode plate is at a distance 0.5 units from the electrode (aperture screen). When (3) is limited to $k=3$, the potential distribution in the aperture is found to be in agreement with the values obtained from electrolytic tank measurements. From the general expression of the axial potential distribution between the cathode and electrode, the author concludes that a very close approximation to the axial potential distribution can be represented by a polynomial of the form

$$(4)\quad \varphi(0, z) = \sum_{k=1}^{3} A_k(z+\zeta)^{2k+1} \quad (-\zeta \leqq z \leqq 0).$$

The coefficients A_k are tabulated for values of ζ ranging from 0.5 to 1. Finally an approximate formula for φ is given for $z \geqq 0$. *N. Chako* (Flushing, N.Y.)

5402:

Hain, K.; and Lüst, R. Zur Stabilität zylindersymmetrischer Plasmakonfigurationen mit Volumenströmen. Z. Naturf. **13a** (1958), 936–940.

The stability of a cylindrically symmetrical plasma configuration possessing volume currents is examined theoretically. By the method of small perturbations the problem is cast into the form of a single second order differential equation and the eigenvalues are computed numerically for a specific current distribution that is strong on the axis and gradually tapers off with radial distance. The analysis appears to show that the configuration is unstable especially for the longer wavelengths. *C. H. Papas* (Pasadena, Calif.)

5403:

Broyles, A. A. Calculation of fields on plasma ions by collective coordinates. Z. Physik **151** (1958), 187–201.

A new calculation is given for the probability distribution of the electric field acting on an ion in a plasma. The electrons are replaced by a uniform negative charge distribution. The potential between ions is cut off by replacing it by zero beyond a certain distance. *L. Van Hove* (Utrecht)

5404:

Kichenassamy, S. Sur le champ électromagnétique singulier en théorie de Born-Infeld. C. R. Acad. Sci. Paris **248** (1959), 3690–3692.

It is shown that in the singular case which is characterized by $B^2=E^2$ and $\mathbf{B}\cdot\mathbf{E}=0$ (E and B are the electric field and the magnetic induction) the Born-Infeld theory gives the same results as the Maxwell's theory. *L. Infeld* (Warsaw)

5405:

Feinberg, E. L. Propagation of radio waves along an inhomogeneous surface. Nuovo Cimento (10) **11** (1959), supplemento 60–91.

This paper gives an excellent survey of the approximate theories dealing with the propagation of electromagnetic waves over a flat or curved earth the electrical parameters of which change from point to point. The first part summarizes work done in the U.S.S.R. which has been published in Russian between 1944 and 1950. The usual treatment of the problem starts from an application of Green's integral theorem to both the unknown wave function solving the problem, E say, and the corresponding wave function for a homogeneous earth. This procedure involves an integral relation into which enter E and its derivative $\partial E/\partial n$ normal to the earth's surface. However, this derivative can be reduced to E itself by applying the Leontovich's boundary condition of the form $\partial E/\partial n=-\lambda E$, the applicability of which is discussed in detail. The resulting two-dimensional integral equation for the distribution of E over the earth's surface can further be reduced to a one-dimensional equation with the aid of a saddlepoint approximation. The latter is sufficiently accurate under very general conditions which depend on the geometry of Fresnel zones and of the boundaries separating domains of different values of the electrical parameters.

The final one-dimensional integral equation for a flat earth can be represented as follows in terms of the attenuation factors (with respect to free-space propagation) $W(D)$ and $W_0(D)$ for the unknown wave function and the homogeneous-earth solution respectively:

$$(1)\quad \begin{aligned} W(D) = W_0(D)-i(D/\pi)^{1/2}\int_0^D [\{s(x')\}^{1/2}-s_0^{1/2}]\\ \times W(x')W_0(D-x')\{x'(D-x')\}^{-1/2}dx';\end{aligned}$$

D here represents the distance along the earth's surface from the source to the point of observation (both assumed here on the ground), $s(x)$ the value at a distance x from the source (on the path connecting the latter with the observation point) of the parameter accounting for the electrical constants of the soil, s_0 the corresponding parameter for a special homogeneous earth. The solution of this integral equation of Volterra type is independent of the soil conditions in the region beyond the observation

point (away from the source). This property permits the solution by steps in the case of an earth's surface consisting of adjacent homogeneous sections. In fact, the W function in the nth section (with parameter S_n) then follows at once from those in the preceding sections by applying (1) for $s_0 = s_n$.

Explicit expressions are given for the special cases of two or three homogeneous sections. Approximations holding if one or more of these sections are large or small (measured in units of Sommerfeld's numerical distance) are also discussed. The final expressions clearly show well-known properties such as the "recovery effect" (field amplitude increasing with distance in a region of increasing conductivity), coastal refraction (change of propagation direction for waves passing a boundary between two homogeneous sections, e.g., a coastline separating sea and land), and the dominating influence of the soil properties near the source and near the observation point. The connection with the related problem concerning the effect of mountainous terrain profiles is also mentioned.

The last section deals with the spherical-earth case which can also be described with the aid of an integral equation of type (1). However, the homogeneous-earth solution W_0 has to be represented here by an infinite series, instead of the error function for the Sommerfeld approximation for a flat earth. Furutsu's treatment of the spherical case [J. Radio Res. Lab. Japan **2** (1955), 345] could be simplified considerably. It shows, e.g., that the dominating influence of the surroundings of the source and receiving point does disappear if the effect of the earth's curvature is well established.

We mention a few serious printing errors. In the inequality $(2-3)$ the sign $\ll$ has to be replaced by $\gg$; in equation (3.14) x and q should read ξ and q_0; in the legend of fig. 3 $D = x_0 + x_A$ has to be replaced by $D' = x_0 + x_A$.

H. Bremmer (Eindhoven)

5406:

De Socio, Marialuisa. Sulla propagazione di onde non sinusoidali in un gas ionizzato soggetto ad un campo magnetico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **92** (1957/58), 243–255.

The propagation of a plane, non-sinusoidal, electromagnetic wave in an ionized gas subjected to a constant magnetic field H_0 is considered for the case where the direction of propagation is parallel to H_0 and for the case where it is perpendicular. The behavior of the field in the neighborhood of the wave-front is studied.

C. H. Papas (Pasadena, Calif.)

5407:

De Socio, Marialuisa. Sulla propagazione obliqua di onde non sinusoidali in un gas ionizzato soggetto ad un campo magnetico. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **92** (1957/58), 383–394.

The propagation of a plane, non-sinusoidal wave in an ionized gas subjected to a magnetic field is studied for the general case where the direction of propagation is oblique. The behavior of the field and its derivative at the wave-front is examined. *C. H. Papas* (Pasadena, Calif.)

5408:

Ufimtsev, P. Ia. Approximate calculation of the diffraction of plane electromagnetic waves by certain metal objects. II. The diffraction by a disk and by a finite cylinder. Soviet Physics. Tech. Phys. **28** (3) (1958), 2386–2396 (2604–2616 Ž. Tehn. Fiz.).

The author's previously reported method [Ž. Tehn. Fiz. **27** (1957), 1840–1849; MR **19**, 1012] is used to calculate the diffraction by a circular disk and by a finite cylinder, both ideally conducting.

For the disk the uniform, or regular, component is found by quadratures under the distant-field assumption; in particular, for an observer on the line joining source and disk the field is the sum of spherical waves diverging from the two points of intersection of the edge of the disk with the plane determined by the normal to the disk through its center and the normal to the incident wave (plane of incidence). The non-uniform component, containing the contribution of the edge, is by definition the sum of the fields of half-planes tangent to the disk at each point of its edge. These fields are calculated as in the earlier paper for an observer in the plane of incidence, so that the only points which are actually taken into account are the two "luminous" points just mentioned. The solution thus obtained for the total scattered field satisfies the principle of reciprocity, but contains an unwanted discontinuity for certain particular cases. In another paper [Soviet Physics. Tech. Phys. **28** (3) 1958, 549–556; MR **20** #7500b] the author has removed this defect by taking secondary diffraction into account.

In the case of the cylinder (of radius a and length l), the non-uniform component is assumed to consist of the contributions from rectangular dihedral angles at the three luminous points determined by the section of the cylinder in the plane of incidence. It is consequently expressed in terms of the scattering functions for a wedge derived in the original paper, and the resulting scattered field is valid only for an observer in the plane of incidence. Other approximations restrict the validity of the results to the region $ka \gg 1$, $kl \gg 1$. Whatever the limitations on the parameters, the solution of this difficult problem is a noteworthy achievement. *R. N. Goss* (San Diego, Calif.)

5409:

Radlow, James. Diffraction of a dipole field by a unidirectionally conducting semi-infinite screen. Quart. Appl. Math. **17** (1959), 113–128.

The analysis is based on the formulation of the diffraction problem so it may be readily solved by a Wiener-Hopf technique. The solution is facilitated by taking the transform of the differential equation before applying the boundary conditions. Since the paper is dealing with a dipole perpendicular to a uni-directional conducting half-plane, two successive transforms of the wave equation are required. The double transform of the electric field is then expressed as an unknown vector function of the transform variables. Then on using Maxwell's equations and applying the boundary and "jump" conditions on the screen, it is shown that the problem reduces to a single transform equation which involves two unknown complex functions. Using a function-theoretical approach, the solution is derived and the resulting fields are expressed as inverse Laplace transforms. The principal part of these are integrals, which were introduced by H. M. MacDonald [Proc. London Math. Soc. (2) **14** (1915), 410–427] and occur in the case of a perfectly conducting

half-plane. The remaining or correction terms can be transformed into real integrals.

It would appear that the method could be readily generalized to other dipole orientations and to half-planes with more realistic boundary conditions.

J. R. Wait (Boulder, Colo.)

5410:

Begiashvili, G. A.; and Gedalin, E. V. Motion of a charged particle in an anisotropic medium. Soviet Physics. JETP **35** (8) (1959), 1059–1061 (1513–1517 Ž. Eksper. Teoret. Fiz.).

In this paper the authors derive the general expression for the total energy loss per unit path length suffered by a point charge moving with a uniform velocity through a medium characterized by a complex dielectric and permeability tensor. The expression is evaluated in two particular cases for an optically active uniaxial crystal: (1) the charge moves along the optical axis; (2) the charge moves normal to the optical axis.

N. L. Balazs (Princeton, N.J.)

5411:

Goertz, Adalbert. Zur Theorie der diffusen Reflexion und Transmission beim Vorliegen elastischer Vielfachstreuung. Z. Physik **155** (1959), 263–274.

This paper concerns stationary states in scattering problems depending on the following integro-differential equation of Bothe

$$(1\text{–}3)\quad \cos\theta\,\frac{\partial F(x,\theta)}{\partial\theta} = -(\mu_0+N\sigma)F(x,\theta) + N\int\sigma_s(\theta)F(x,\theta')d\Omega'.$$

The function F is defined as the total flux, through a plane perpendicular to the x-axis, of particles moving within a unit solid angle making an angle θ with this axis; the velocity distribution of the particles is assumed as symmetrical around this axis. The medium is assumed to absorb particles (absorption coefficient μ_0), as well as to effect elastic scatterings (N being the density of the scattering centres). Further, $\sigma=\sigma_a+\sigma_s$ represents the total effective cross-section (due to both absorption and scattering), $\sigma_s(\vartheta)$ the differential cross-section for a special scattering angle ϑ, and $d\Omega'$ a solid-angle element in a direction making an angle θ' with the x-axis.

The article discusses the reduction of (1–3) to a pair of ordinary differential equations, viz.,

$$(1.11)\quad \begin{aligned} \frac{dI(x)}{dx} &= -(K+S)I(x)+SJ(x),\\ -\frac{dJ(x)}{dx} &= -(K+S)J(x)+SI(x), \end{aligned}$$

for two quantities $I(x)$ and $J(x)$. The latter are the flux of particles having a forward or a backward velocity component, respectively, in the x direction. The derivation of (1.11) is based on the approximation $\cos\theta\sim 1$ or -1 for these two groups of particles, and on reasonable assumptions concerning the two different expansions of $F(x,\theta)$ into Legendre functions of odd order for the two ranges $0<\theta<\pi/2$ and $\pi/2<\theta<\pi$.

The coefficients K and S depend on μ_0, N, σ_a and σ_s. The general discussion of (1.11) leads to explicit expressions (3.8b) and (3.9b) for the reflection and the transmission coefficient of a slab producing absorption and scattering of the described form. The final results show the role of an effective absorption coefficient $b=\{K(K+2S)\}^{1/2}$ which replaces the value μ_0 holding for pure absorption without scattering.

H. Bremmer (Eindhoven)

5412:

Lyudbarskii, G. Ya.; and Povzner, A. Ya. On the theory of wave propagation in variable-cross-section waveguides. Soviet Physics. Tech. Phys. **29** (4) (1959), 146–154 (170–179 Ž. Tehn. Fiz.).

The propagation of an acoustic (scalar) wave in a waveguide of variable cross section is studied. Essentially the technique is a generalization of the W.K.B. method. Rather than invoking the usual assumption that the angle α between an arbitrary tangent plane to the waveguide surface and the z or waveguide axis is small, the authors allow α to be of any magnitude but its rate of change is to be small. That is, $\lambda(d\alpha/dz)\ll 1$ where λ is the wavelength.

The proposed method is applied to an example which possesses axial symmetry. The waveguide consists of two coaxial cones I and III connected by a portion II of length L of variable cross section. In the variable section the solution has the form of a W.K.B. series similar to that developped by Bremmer [Comm. Pure Appl. Math. 4 (1951), 105–115; MR **13**, 462]. In the conical regions, the solutions are written in terms of spherical wave functions involving Hankel functions of complex order. By matching fields across suitable spherical surfaces, relations are obtained which connect the various coefficients.

The analysis is mainly of a formal nature.

J. R. Wait (Boulder, Colo.)

5413:

Baret, Hélène. Diagrammes de directivité d'un dipôle vertical en présence d'un parasite cylindrique parallèle dans le plan horizontal. Ann. Télécommun. **14** (1959), 220–235.

"Le calcul de l'effet d'un dipôle perturbateur mince sur un dipôle parallèle également mince est connu. L'auteur le généralise au cas où les deux dipôles sont des cylindres plus ou moins épais et non identiques, en appliquant la méthode de Hallen. Les diagrammes calculés par cette méthode sont en assez bonne concordance avec ceux calculés par la méthode classique et ceux relevés sur modèles réduits. Les résultats obtenus à partir des premiers diagrammes concordent également assez bien avec ceux (théoriques) obtenus par King et ceux (expérimentaux) obtenus par Owen." *Résumé de l'auteur*

5414:

Teichmann, Horst. Das Gausssche Prinzip des kleinsten Zwanges und die Möglichkeiten seiner Anwendungen auf elektrotechnische Probleme. Arch. Elektrotech. **44** (1959), 275–278.

5415:

★Stigant, S. Austen. The elements of determinants, matrices and tensors for engineers. Macdonald, London, 1959. xi+433 pp. 60s.

The chief object of this book is to present to electrical

engineers an elementary study of determinants, matrices and tensors. The theory is interlinked closely with that of simultaneous equations. There are many examples with detailed solutions and the work is very clearly written and typeset. Four appendices on tensors go a little beyond the elementary standard of the main text.

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 5386.

5416:

Gleyzal, A. On the determination of certain thermodynamic and physical quantities. Quart. Appl. Math. **17** (1959), 318–320.

Etant donné un phénomène physique, où une quantité z est une fonction des 2 variables x et y, avec $dz = Fdx + Gdy$, l'Auteur montre que si y et G peuvent être mesurés directement, les quantités x et F peuvent alors être déterminées en fonction de y et de G, pourvu que l'on connaisse la famille des courbes $F = \text{const}$, ainsi que 2 courbes $x = \text{const}$. Une application de ce résultat à la détermination des isothermes et des adiabatiques dans le plan, p, V, est donnée. *R. Gerber* (Grenoble)

5417:

★Yachter, M.; and Mayer, E. Conduction of heat. Turbulent flows and heat transfer (edited by C. C. Lin), pp. 254–287. High Speed Aerodynamics and Jet Propulsion, Vol. V. Princeton University Press, Princeton, N.J., 1959. xv + 549 pp. (2 plates) $15.00.

In spite of its title, this is not a short treatise on the conduction of heat but a short survey article which discusses selected problems of direct interest in the design of rocket and jet combustion chambers and nozzles, of skins of high-speed aircraft, turbine blades, etc.

J. Kestin (Providence, R.I.)

5418:

★Deissler, Robert G.; and Sabersky, R. H. Convective heat transfer and friction in flow of liquids. Turbulent flows and heat transfer (edited by C. C. Lin), pp. 288–338. High Speed Aerodynamics and Jet Propulsion, Vol. V. Princeton University Press, Princeton, N.J., 1959. xv + 549 pp. (2 plates) $15.00.

A careful survey of problems on the title subject including a separate chapter on problems in boiling heat transfer by the second author.

J. Kestin (Providence, R.I.)

5419:

★van Driest, E. R. Convective heat transfer in gases. Turbulent flows and heat transfer (edited by C. C. Lin), pp. 339–427. High Speed Aerodynamics and Jet Propulsion, Vol. V. Princeton University Press, Princeton, N.J., 1959. xv + 549 pp. (2 plates) $15.00.

A careful review of current problems involving forced convection from gases with special attention paid to compressible flows. A very useful survey for those interested in the subject. *J. Kestin* (Providence, R.I.)

5420:

★Penner, S. S. Physical basis of thermal radiation. Turbulent flows and heat transfer (edited by C. C. Lin), pp. 489–501. High Speed Aerodynamics and Jet Propulsion, Vol. V. Princeton University Press, Princeton, N.J., 1959. xv + 549 pp. (2 plates) $15.00.

The article gives a concise description of the phenomena involved in the calculation of emitted and absorbed radiant energy from equilibrium flames. The calculation is reduced to the evaluation of total emissivity in the form of an integral over wave-number of a function involving the spectral absorption coefficient P_ω known from spectroscopic measurements.

A brief but clear summary and guide through recent literature. *J. Kestin* (Providence, R.I.)

5421:

Golubenkov, V. N. Thermal convection in a rotating circular pipe with a constant temperature gradient (compressible fluid). J. Appl. Math. Mech. **22** (1958), 1205–1207 (840–841 Prikl. Mat. Meh.).

5422:

Leont'ev, A. I. The analytical investigation of gas flow in a cylindrical tube with heat transfer. Inž.-Fiz. Ž. **1** (1958), no. 5, 46–55. (Russian)

5423:

Močalin, A. I. The application of the Dirac function to the solution of heat conduction problems. Inž.-Fiz. Ž. **1** (1958), no. 5, 76–83. (Russian)

5424:

Zaidel', R. M.; Ryzhov, O. S.; and Andriankin, E. I. The propagation of an approximately spherical heat front. Soviet Physics. Dokl. **124** (4) (1959), 65–67 (57–59 Dokl. Akad. Nauk SSSR).

5425:

Bailey, H. R. Heat conduction from a cylindrical source with increasing radius. Quart. Appl. Math. **17** (1959), 255–261.

Let $T(r, t)$ denote temperatures in an infinite solid initially at temperature zero when heat is generated uniformly over an expanding cylindrical surface $r = r_P(t)$. The time rate of generation of heat is assumed to be proportional to $r_P'(t)$, the rate of increase of the radius of the cylinder. In the two cases $r_P = Ut$ and $r_P^2 = 2Vt$, where U and V are constants, the author writes formulas for $T(r, t)$ in terms of integrals involving exponential and Bessel functions and then makes an analysis of the limit of $T(r, t)$ as $t \to \infty$. *R. V. Churchill* (Ann Arbor, Mich.)

5426:

Saelman, B. Integration of some thermal differential equations. J. Aero/Space Sci. **26** (1959), 754–755.

5427:

Struble, Raimond A. A study of the interior ballistic equations. Arch. Rational Mech. Anal. **3**, 397–416 (1959).

The author studies some mathematical characteristics of the solution of certain nonlinear ordinary differential equations arising in the theory of interior ballistics. It examines critically the generally popular choice of unity for burning exponent. This paper avoids technological questions and therefore avoids quantitative estimates of relative effects of simplifying assumptions. It accepts zero as starting shot pressure, and a quadratic form function maintained throughout. It ignores all phenomena subsequent to burning through of the web, all effects of friction, energy dissipation, pressure waves in the gas, etc.

A. A. Bennett (Providence, R.I.)

5428:

Kapur, J. N. A note on the analytical solution of the equations of internal ballistics for a tapered-bore gun. Proc. Nat. Inst. Sci. India. Part A **24** (1958), 319–322.

J. Corner attempted to show that by a simple transformation the ballistic equations for a tapered-bore gun could be reduced to those for an orthodox gun [*Theory of internal ballistics of guns*, Wiley, New York, 1950; MR **12**, 213]. This author claims that the reduction then given was not complete. For a conical-bored gun of small taper, J. K. Jain proposed two simplifying assumptions to effect such a reduction, the first being that the variable area of cross-section can be replaced by a mean value. In this paper, the author shows that this first assumption suffices.

A. A. Bennett (Providence, R.I.)

5429:

Kapur, J. N. The general theory of moderated charges. II. Proc. Nat. Inst. Sci. India. Part A **24** (1958), 323–329.

The present paper continues the discussion started under this title in same Proc. **23** (1957), 73–92 [MR **19**, 808]. Except under simplifying assumptions no homogeneous charge of constant characteristics can be equivalent to a moderated charge (of several layers) except after all-burnt. Here are investigated restrictions adequate for such equivalence.

A. A. Bennett (Providence, R.I.)

5430:

Kapur, J. N. Effects of variations of loading conditions on internal ballistics. Proc. Nat. Inst. Sci. India. Part A **25** (1959), 1–21.

To determine the variations in function values of the three major internal ballistic functions in different phases one has nominally to consider 90 separate tables for partial derivatives. Using ballistic similitudes available under three familiar simplifying assumptions the author offers twelve compact tables sufficing for the whole problem.

A. A. Bennett (Providence, R.I.)

5431:

Kapur, J. N. The internal ballistics of a supergun. Appl. Sci. Res. A **8** (1959), 393–402.

The notion of a supergun (one with successively impelling powder chambers) has repeatedly occurred to inventors. The ballistic equations for such a gun were given by Jain and Sodha [Appl. Sci. Res. A **7** (1958), 369–374; MR **20** #664] under the (implausible) assumption that the propellant in the rth powder chamber starts burning at the instant that the powder in the previous chamber has been completely consumed. This author shows that the equations are then exactly those for a moderated charge (composed of identical grains each built up of layers). Such a gun is theoretically inefficient. A more generally inclusive theory is here also discussed.

A. A. Bennett (Providence, R.I.)

5432:

Tawakley, V. B. Effect of bore resistance on internal ballistics of guns for composite charge consisting of n component charges. Proc. Nat. Inst. Sci. India. Part A **25** (1959), 22–45.

Little is known concerning bore resistance other than for initial engraving of driving band. Such resistance has usually been either ignored or supposedly accounted for by some averaging correction factor or additive term (as by Corner). This paper extends Corner's method to the case of a general composite charge (of n charges of different shapes, sizes and compositions).

A. A. Bennett (Providence, R.I.)

5433:

Lifshitz, I. M.; and Slezov, V. V. Kinetics of diffusive decomposition of supersaturated solid solutions. Soviet Physics. JETP **35** (8) (1959), 331–339 (479–492 Ž. Eksper. Teoret. Fiz.).

Suppose a spherical solute particle of radius R is immersed in a supersaturated solution of concentration C. The equilibrium concentration at the surface is $C_\infty+\alpha R^{-1}$, where C_∞ is the concentration of a saturated solution and α is a constant related to the surface tension. According as $C \gtreqless C_\infty+\alpha R^{-1}$ the particle will grow, not change or decay. Thus there is an equilibrium radius $R_{cr}=\alpha(C-C_\infty)^{-1}$ such that particles with $R<R_{cr}$ decay and those with $R>R_{cr}$ grow. Consequently the particle size distribution function $f(R, t)$, which is the number of particles per unit volume per unit R, will change with time. The concentration $C(t)$ of the solution will also change as some particles dissolve and as some solute condenses on others. The problem treated is that of finding $f(R, t)$ and $C(t)$—both assumed to be uniform in space—for large t given them initially. The basic equations are the quasi-static equation for the rate of growth of the particle radius, and the conservation of mass equation.

$$(1)\qquad \frac{dR}{dt} = DR^{-1}[C(t)-C_\infty-\alpha R^{-1}],$$

$$(2)\qquad \frac{4\pi}{3}\int_0^\infty R^3 f(R, t)dR+C(t) = \text{const.}$$

In addition if $R(t, R_0)$ is the solution of (1) with $R(0)=R_0$, the definition of f yields

$$(3)\qquad f[R(t, R_0), t] = f[R_0, 0].$$

The asymptotic form of the solution of these equations for f, C and R is obtained as $t\to\infty$. The result is used to discuss the corresponding problem in which a plane boundary is present.

J. B. Keller (New York, N.Y.)

QUANTUM MECHANICS

See also 4901, 4995, 5483, 5484.

5434:

Chou, Kuang-Chao; and Zastavenko, L. G. The Shapiro integral transformation. Soviet Physics. JETP **35** (8) (1959), 990–995 (1417–1425 Ž. Eksper. Teoret. Fiz.).

The authors extend to the case of non-vanishing spin the result of Shapiro [Dokl. Akad. Nauk SSSR **106** (1956), 647–649; MR **17**, 1181]. They obtain in highly explicit form the direct integral decomposition of the restriction to the homogeneous Lorentz group of the natural (irreducibly unitary) representation of the inhomogeneous group in the (Hilbert) space of wave functions for a particle of arbitrary mass and spin. *I. E. Segal* (Chicago, Ill.)

5435:

Henneberger, Walter C. Exakte Lösungen für die Dipolstrahlung des harmonischen Oszillators. Z. Physik **155** (1959), 296–312.

The Schrödinger equation for a charged, non-relativistic particle without spin in interaction with the electromagnetic field can be solved in dipole approximation. Using the solution given by the reviewer [Danske Vid. Selsk. Mat.-Fys. Medd. **26** (1951), no. 15; MR **13**, 807] the author derives the following results for the case that the particle is harmonically bound. (i) There is a solution (rigorous within dipole approximation) in which the particle wave function is a Gaussian wave packet whose center obeys the classical equation for a damped oscillator. (ii) There is another solution in which the initial field energy is zero while the initial particle energy is $h\nu$; this energy then decreases exponentially with the same lifetime that one usually derives by means of perturbation theory. (iii) For the latter solution the frequency distribution of the emitted radiation is calculated explicitly and found to have the expected natural line width. (iv) Finally, solutions are constructed in which the initial energies of the particle are $2h\nu$, $3h\nu$, $\cdots$. They decay with the same lifetime as the first excited state (ii). *N. G. van Kampen* (Utrecht)

5436:

★Rubin, Herman. On the foundations of quantum mechanics. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 333–340. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

A partially heuristic comparison of the mathematical status of the following approaches to classical quantum mechanics: 'A, the Hilbert space formulation with unitary transition operators; B, the matrix-transition-probability-amplitude formulation; and C, the phase-space formulation'. *I. E. Segal* (Chicago, Ill.)

5437:

Feenberg, Eugene. Analysis of the Schrödinger energy series. Ann. Physics **3** (1958), 292–303.

"The rate of convergence of the Brillouin-Wigner and Schrödinger energy series can be modified by using elements of freedom present in the formulation of the perturbation problem. Two such elements of freedom are (a) a uniform displacement of the zeroth order energy spectrum and (b) a uniform change of scale in the spacing of the zeroth order energy levels. Transformation (a) is used to generate the Schrödinger energy series from the corresponding Brillouin-Wigner series. The effect of transformation (b) on the former series is worked out and exhibited explicitly.

"The problem of defining a criterion for determining the scale factor is bypassed by the observation that continued fraction approximants to the Schrödinger energy series are invariant under transformation (b). Explicit formulas are given for the first three invariant forms." (Author's summary) *M. Cini* (Rome)

5438:

Lochak, Georges. Quelques problèmes sur le groupe des rotations et la toupie quantique. Cahiers de Phys. **13** (1959), 41–80.

This is a reinvestigation of the problem of the quantum dynamics of the spinning top (rigid body). The author's special concern is the justification of admitting half integer angular momentum quantum numbers j. Such values are admissible, the author shows, by imposing on the wave function ψ the property of being continuous and one-valued on the surface S_4 of a 4-dimensional unit sphere (representing the 3-dimensional rotation group). The only new result presented is a discussion of the classical limit, and a demonstration of the nonvalidity of Ehrenfest's theorem. Some statements made in § 16, concerning the application of these results to molecules, are incorrect and misleading. *F. Villars* (Cambridge, Mass.)

5439:

Fujiwara, Izuru. On the space-time formulation of non-relativistic quantum mechanics. Progr. Theoret. Phys. **21** (1959), 902–918.

This is an attempt to develop the form of the propagation function $K(xt, x't')$ of non-relativistic quantum mechanics without use of the canonical formalism, starting from the classical action function

$$S(xt, x't') = \int_{t'}^{t} d\tau L(q, \dot{q}, \tau).$$

The author develops an alternative to the Feynman solution [Rev. Mod. Phys. **20** (1948), 367–387; MR **10**, 224] of this problem, which avoids the complicated path integrals: K is first approximated by the semi-classical kernel K_c:

$$K_c(xt, x't') = ((i/\hbar)\partial^2 S/\partial x \partial x')^{1/2} \exp((i/\hbar)S)$$

and the corrections to K_c found by invoking the composition rule

$$K(xt, x't') = \int dx'' K(xt, x''t'')K(x''t'', x't').$$

F. Villars (Cambridge, Mass.)

5440:

Moravcsik, Michael J. Coulomb effects in boson emission. Phys. Rev. (2) **114** (1959), 621–625.

This paper extends the conventional treatment of electromagnetic corrections in the production of spinless particles by including relativity and finite-size effects. For a point-nucleus the use of relativistic kinematics for the emitted particle generalizes the Coulomb penetration factor by a simple modification of the ordinary Coulomb wave function. Finite size effects are calculated for a model in which the charge is distributed on the surface of a sphere instead of being concentrated at a point.

S. Bludman (Berkeley, Calif.)

5441:

★**Любарский, Г. Я. Теория групп и ее применение в физике. [Lyubarskiĭ, G. Ya. Group theory and its application to physics.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 354 pp. 13.20 rubles.

This book invites comparison with the book of Wigner reviewed below. The twenty-five years between their publication dates have made it so obvious that group theory is an important tool in theoretical physics that the latter day author need not waste his didactic zeal on the most elementary aspects of the subject; his reader may not yet know the subject but he knows matrix algebra and accepts as well known that abstract ideas of groups, operators, cosets, etc., are worth learning for the sake of their applications to physics. In a book of about the same number of pages the author can cover many more applications and give a good deal more technical detail. The book is divided into seventeen chapters and an appendix. The first chapter defines the notions of group, subgroup, coset, isomorphism and homomorphism. The second defines some examples of groups: the symmetric group, the orthogonal and euclidean groups of three-dimensional space, the point groups and space groups in three dimensions. Chapters three and four are devoted to the theory of group representations. The results are mostly stated and proved for finite groups. Chapter 5 gives some representations of some of the groups discussed in Chapter 2. Chapter 6 applies the results to the theory of small vibrations, Chapter 7 to the theory of phase transitions of the second kind, and Chapter 8 to crystals. Chapter 9 discusses the extension of the results of the representation theory of finite groups to infinite groups. Here the reader is referred elsewhere for precise definitions and proofs of many of the main facts. Chapters 10 and 11 treat the representations of the three-dimensional orthogonal group and the vector addition and Racah coefficients respectively. Chapter 12 connects the symmetry group of the Hamiltonian to integrals of motion for the Schrödinger equation. Chapter 13 considers the transformation properties of field equations invariant under Euclidean transformation. Chapter 14 deals with selection rules in the absorption and scattering of light. Chapter 15 is devoted to the brief treatment of the representations of the Lorentz group. Chapter 16 gives an outline of the Gelfand-Yaglom theory of invariant wave equations. Chapter 17 is a brief indication of the application of the theory of representations of the rotation group to the theory of nuclear reactions. There is an appendix containing character tables to the symmetric, point, and space groups and a bibliography. To the reviewer the book seems a useful one, except that the treatment of relativistic quantum mechanics does not go far enough to give the reader much of an idea of the real problems in that part of the subject.

A. S. Wightman (Princeton, N.J.)

5442:

★**Wigner, Eugene P. Group theory: And its application to the quantum mechanics of atomic spectra.** Expanded and improved ed. Translated from the German by J. J. Griffin. Pure and Applied Physics. Vol. 5. Academic Press, New York-London, 1959. xi+372 pp. $8.80.

This book is a translation, with additions, of one of the three pioneering works on the application of group theory to quantum mechanics. The additions are chapters on space inversion, the Racah coefficients, and the classical limits of the vector addition and Racah coefficients. The book retains its flavor; it was always the easiest of the three for the physicist neophyte because of the elementary character of its mathematical exposition. It begins with three chapters on matrix algebra, and three on elementary quantum mechanics. There follow three chapters on the theory of finite groups and their representations and a chapter on continuous groups. In chapters 11, 12 and 16 the symmetry of the eigenvalue problem of the Schrödinger equation is connected with a group representation. Chapters 13, 14 and 15 are devoted to the symmetric group and the three-dimensional rotation group. Chapters 16 through 23 and 25 give the application to atomic spectra including such subjects as: selection rules and splitting of spectral lines, partial determination of eigenfunctions from their transformation properties, selection and intensity rules with spin and the building-up principle. Although the book contains no discussion of applications of group theory to relativistic problems and today's students, better trained in the theory of vector spaces and linear transformations, may prefer a more geometrical mode of exposition there is little doubt that it will continue to be useful for years to come.

A. S. Wightman (Princeton, N.J.)

5443:

Winter, Rolf G. Klein paradox for the Klein-Gordon equation. Amer. J. Phys. **27** (1959), 355–358.

This is a study of the well known Klein paradox [O. Klein, Z. Physik **53** (1929), 157–165] for the case of spin zero particles. The penetrability of a potential barrier of height $V>2mc^2$ is calculated, and the well known relation $R-T=1$ is displayed. (R=reflexion coefficient, T= transmission coefficient of the barrier.) In contrast to the case of spin 1/2 particles, it is shown that the results may be interpreted directly, and are consistent with energy and charge conservation.

F. Villars (Cambridge, Mass.)

5444:

Buchdahl, H. A. On extended conformal transformations of spinors and spinor equations. Nuovo Cimento (10) **11** (1959), 496–506. (Italian summary)

This paper deals with the invariance, under the extended conformal group, of relativistic equations for fields of spin $S>0$ and mass $K\neq 0$. By the extended conformal group is meant that group of transformations on the space-time metric taking $\eta_{\mu\nu}\to\lambda\eta_{\mu\nu}$, where λ is an arbitrary positive function of the coordinates. For $S\geqq 1$ the field equations are supplemented by subsidiary conditions assuring that no states of spin $<S$ are admitted.

These subsidiary conditions are not conformally invariant, so that the author concludes that the Dirac equation ($S=\frac{1}{2}$) with k a conformal covariant (rather than invariant) is the only conformal relativistic wave equation. The treatment is unquantized and does not include any form of interaction. *S. Bludman* (Berkeley, Calif.)

5445:

Singh, V. On the interaction Hamiltonian of symmetric pseudoscalar meson theory. Nuovo Cimento (10) 11 (1959), 800–804. (Italian summary)

The equivalence of $\mathbf{H}_{PS}$ and $\mathbf{H}_{PV}$, the pseudoscalar and pseudovector couplings of pseudoscalar pions to nucleons, is investigated without using the conventional Dyson canonical transformation. Working in the interaction representation, from $\mathbf{H}_{PS}$ a quantity $\mathbf{H}$ is obtained which is, for scattering states, a Hamiltonian. $\mathbf{H}$ is the sum of $\mathbf{H}_{PV}$, a pair coupling term, and a nonlocal interaction.

S. Bludman (Berkeley, Calif.)

5446:

Candlin, D. J. The supplementary condition in quantum electrodynamics. Nuovo Cimento (10) 12 (1959), 54–62. (Italian summary)

The supplementary condition in quantum electrodynamics has been a vexing question which, although satisfactorily treated by the method of the indefinite metric [S. N. Gupta, Proc. Phys. Soc. Sect. A **63** (1951), 681–691; MR **12**, 67], is still felt to be in need of further clarification. In this paper, the author points out that the supplementary condition presents features similar to the treatment of centre of mass motion in many body problems. These are, namely, the difficulties that occur when one restricts one's attention to a linear manifold spanned by the (improper) eigenfunctions belonging to a point in the continuous spectrum of a quite well defined operator. All the states are naturally non-normalizable, but a suitable redefinition of norm is easily given. One method used in the many body problem is to replace the condition $P\psi=0$ by $(P-iM\omega R)\psi=0$, with ω an arbitrary constant. A corresponding replacement of the supplementary condition can be made, although it is too long to quote here. {In the condition stated in equation (13) there is not longer an arbitrary constant corresponding to the ω above.} In addition to overcoming the normalization difficulty it is also necessary to show that the results are invariant under the Lorentz group. Although the conditions are not manifestly covariant they are shown by the author to be so. *C. A. Hurst* (Adelaide)

5447:

Hara, Osamu. A study of charge independence in terms of Kaluza's five dimensional theory. Progr. Theoret. Phys. **21** (1959), 919–937.

"An attempt is made to give a foundation to the charge independence based on the five dimensional theory proposed by Kaluza. It is shown that his fifth coordinate can be identified with the angular variable of the charge space describing the rotation around its third axis, and that a transformation with respect to this coordinate proposed by Klein as a generalization of the gauge transformation is isomorphic to the rotation in a three dimensional Lorentz space. It is shown that this space has a close relation to the charge space." (Author's summary)

P. W. Higgs (London)

5448:

Uhlmann, Armin. Zur zweiten Quantelung bei symmetrischer Statistik. Wiss. Z. Karl-Marx-Univ. Leipzig. Math.-Nat. Reihe 7 (1957/58), 115–122.

A discussion is given of the properties of the Fock functionals [Z. Phys. **49** (1928), 339–357] defined for bosons. The discussion includes the definition of functional derivatives, second quantization, annihilation and creation operators, and scalar products of Fock functionals.

D. ter Haar (Oxford)

5449:

Nagy, K. L. On an equivalence theorem for integro-differential equations occurring in field theories. Acta Phys. Acad. Sci. Hungar. **10** (1959), 195–198. (Russian summary)

Królikowski and Rzewuski [Nuovo Cimento (10) **2** (1955), 203–219; MR **17**, 334] have earlier proved that an integro-differential equation of the type $D\psi=\int K\psi dx$, where D is a differential operator and the integration on the right extends over 4-dimensional Lorentz space is equivalent to an equation $D\psi=\int A\psi d\sigma$. Here σ is a space-like surface. The author gives a new proof.

A. Salam (London)

5450:

Okai, Sueji. On the relations between spin polarizations and distorted wave theory of the direct reactions. Progr. Theoret. Phys. **22** (1959), 89–100.

This is an investigation of the effect of wave distortion on the polarization of the emitted particle in nuclear stripping (d, p) reactions and inelastic scattering processes. Assuming in the first case that the neutron is captured in a definite state (l, j), and that the second process is a direct interaction, exciting a single nucleon from state (l, j) to (l', j'), the author presents closed expressions for the polarization of the reaction product in terms of integrals over radial wave functions only. The purpose of the paper is then to show that—under the assumptions of his calculation—both magnitude and sign of polarization are closely related to the wave distortion in initial and final states. *F. Villars* (Cambridge, Mass.)

5451:

Nakamura, Kôsuke; and Soga, Michitoshi. Use of antisymmetrized wave function for deuteron stripping reaction. Progr. Theoret. Phys. **21** (1959), 837–855.

The formalism of second quantization is used to derive an expression for the S-matrix of a nuclear reaction. This is used to discuss deuteron stripping reactions, and a comparison is made with other methods. In order to show the effect of the antisymmetry of the wave function, the authors calculate the angular distribution of protons from the reaction $O^{17}(d, p)O^{18}$.

D. J. Thouless (Birmingham)

5452:

Olszewski, S. On the Coulomb and exchange operators in the free-electron model. Acta Phys. Polon. **18** (1959), 121–132.

"The Coulomb and exchange operators occurring in the

Hartree-Fock equations for molecular orbitals of the linear free-electron model are given for the case of a potential box radius tending to zero in the form of explicit and finite functions of the position of the electron on which the operators actually act. The Hartree-Fock differential-integral equations are then replaced by ordinary differential equations.

This has been done with the reservations that the Coulomb and exchange operators were calculated by assuming a suitably modified expression for the mutual interaction between electrons and an exchange potential of the Slater type averaged over the entire region of the potential box was introduced as the exchange operator. The calculation gave expressions reproducing the energies of the respective states, whereby the approximations introduced are taken into account, plus quantities that are the same for all states. The form proposed for the Coulomb and exchange operators is that obtained from the above calculations, where those parts contributing to the reproduction of the above-mentioned uniform quantities which exceed the energies of the calculated states, or which gave a value of zero, are discarded." (Author's abstract) *A. C. Hurley* (Melbourne)

5453:

★**Löwdin, Per-Olov. Correlation problem in many-electron quantum mechanics. I. Review of different approaches and discussion of some current ideas.** Advances in chemical physics (edited by I. Prigogine), Vol. 2, pp. 207–322. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London; 1959. ix+412 pp. $11.50.

This review deals with the historical development and current status of those treatments of the correlation problem which have proved useful in the electronic theory of atoms and molecules. The starting point of the discussion is the Hartree-Fock approximation which is used as a point of reference for a precise definition of the correlation error (p. 235). Three improvements on this approximation are considered in detail; the method of superposition of configurations, the method of correlated wave functions containing inter-electronic coordinates (r_{ij}), and the method using different orbitals for different spins. It is shown how projection operators may be used to advantage in all three approaches.

Numerical results obtained for the helium atom by different methods are carefully analysed. Here wave functions containing r_{ij} terms provide much the most rapid convergence to the exact solution, but satisfactory accuracy may be achieved by the superposition of a large number of orbital configurations.

For systems containing more than two or three electrons the method of configuration interaction is of prime importance. It is pointed out that the development of modern digital computers has greatly accelerated progress in this field. The physical interpretation of the results of these complex calculations in terms of density matrices and natural spin orbitals is stressed. The unrestricted and extended Hartree-Fock schemes, which employ different orbitals for electrons of different spin, are also considered. Although these schemes provide at best a partial elimination of the correlation error, they have the advantage of retaining much of the physical simplicity of the one electron picture.

The emphasis throughout is on the calculation of strict upper bounds to the energy using explicit wave functions and no mathematical approximations. Thus recent developments of the method of atoms in molecules, which have shed considerable light on the role of electron correlation both in large conjugated molecules and in simpler diatomic systems, are not considered and little attention is paid to treatments of the correlation problem in fields other than atomic and molecular theory, e.g., the reaction matrix approach of Brueckner, and the plasma model of Bohm and Pines. Within these limitations the review is comprehensive and authoritative.

A. C. Hurley (Melbourne)

5454:

★**Yoshizumi, Hiroyuki. Correlation problem in many-electron quantum mechanics. II. Bibliographical survey of the historical development with comments.** Advances in chemical physics (edited by I. Prigogine), Vol. 2, pp. 323–365. Interscience Publishers, Inc., New York; Interscience Publishers Ltd., London; 1959. ix+412 pp. $11.50.

This bibliography with occasional brief comments provides a useful supplement to the previous article. It is more comprehensive in that references to work outside the somewhat limited scope of the main article are included. *A. C. Hurley* (Melbourne)

5455:

Löwdin, Per-Olov; and Rèdei, Lajos. Combined use of the methods of superposition of configurations and correlation factor on the ground states of the helium-like ions. Phys. Rev. (2) **114** (1959), 752–757.

"The accurate solution of the Schrödinger equation by means of the method of correlation factor and the method using superposition of configurations is discussed. For a many-electron system, the total wave function divided by the nodeless function $g=g(r_{12}, r_{13}, r_{23}\cdots)$ may be expanded in a series of Slater determinants. For a two-electron system the expansion becomes very simple: the method is applied to the He-like ions, where rapid convergence is found." (From the authors' summary)

D. F. Mayers (Oxford)

5456:

Ladik, J. The ground state of the hydrogen molecule on the basis of the relativistic quantum mechanics with the aid of the Wang wave function. I. Breit equation of the hydrogen molecule. Calculation of the relativisitic correction terms of the kinetic energy. Acta Phys. Acad. Sci. Hungar. **10** (1959), 271–290. (Russian summary)

The relativistic correction to the kinetic energy of the hydrogen molecule in its ground state is calculated from the reduced Breit equation [Phys. Rev. (2) **36** (1930), 383–397] by using the approximate wave function of Wang [ibid. **31** (1928), 579–586]. The correction is -0.00381 eV, which is about twice as large as that for the ground state of helium.

W. Byers Brown (Manchester)

5457:

Gourdin, Michel. Contribution a l'étude covariante de le diffusion nucléon–nucléon à haute énergie. Ann. Physique **4** (1959), 595–641.

This is a potentially important article. Its main contribution is the demonstration that a covariant meson-theoretical calculation of the nucleon–nucleon interaction can naturally yield a repulsive core and produce a change in sign in the $^1S_0$ phase shift at an energy well compatible with experimental information. The calculation uses the Bethe-Salpeter equation as a starting point, and develops a set of coupled integral equations which is solved both by variational methods and by iteration. A formalism is developed which gives phase shifts in terms of these covariant functions. This formalism is a generalization of the usual method of partial waves used in non-relativistic problems, and utilizes hyperspherical harmonics and Gegenbauer polynomials. The calculations are carried out for both Klein-Gordon and Dirac particles. The paper is an extension of the program by M. Levy [e.g. Phys. Rev. (2) **98** (1955), 1470–1478; MR **17**, 567]. If the approximations made in the present paper could be shown to be good, this paper might have an important influence in the theory of nuclear forces, since it shows that a covariant treatment gives even qualitatively a different result from those of the static theories and can explain things that the static theories have found to be a stumbling block for the last decade. *M. J. Moravcsik* (Livermore, Calif.)

5458:

Neugebauer, Th. Zu dem Problem der Elementarladung und der Mesonenmassen. Acta Phys. Acad. Sci. Hungar. **10** (1959), 327–336. (Russian summary)

Models of gases consisting of electrons and protons are considered in terms of the sums of various interaction energies (coulomb, Fermi, etc.). The energies are then minimized as a function of the interparticle separation. For the case of a gas of electrons and an equal number of protons, this distance is of the order of 6×10^{-8} cm; the model then is said to represent a metal. The author then considers a gas of protons "compensated" with an equal number of heavy negative charges as the model of nuclear matter. By suitably choosing interaction energies he arrives at a separation of 10^{-13} cm. The author emphasizes that his result is arrived at "without any arbitrary nuclear forces such as pi-mesic fields, etc.". *S. Deser* (Waltham, Mass.)

5459:

Nagy, K. Angular correlation between neutrino and gamma quantum in *L*-capture. Acta Phys. Acad. Sci. Hungar. **10** (1959), 199–219. (1 insert) (Russian summary)

5460:

Nagy, K. Mass reversal and the interactions of elementary particles. Acta Phys. Acad. Sci. Hungar. **10** (1959), 441–448. (Russian summary)

The suggestion is made that all interactions are invariant under each of the following mass reversal transformations: (a) $M\Psi(x;\kappa)M^{-1}=\eta\gamma_5\Psi(x;-\kappa)$ for Fermions; (b) $M\Phi(x;\kappa)M^{-1}=\omega\Phi(x;-\kappa)$ for ω bosons; and (c) $N\Psi(x;0)N^{-1}=\eta'\gamma_5\Psi(x;0)$.

Fermions are assumed to be grouped into doublets (p,n), (μ,μ_0) and (e,ν); μ_0 is a neutral fermion having zero rest mass, possibly different from the neutrino, and hyperons are considered to be bound states of nucleons and anti-*K*-mesons. Transformation (a) must be applied to both members of a doublet, but (b) and (c) may be applied to single particles. The forms are exhibited of the various types of interaction Hamiltonians which this principle permits. *C. A. Hurst* (Adelaide)

5461:

Winogradzki, Judith. Sur les grandeurs conservatives des particules de spin 1/2 et de masse nulle. C. R. Acad. Sci. Paris **249** (1959), 1087–1089.

5462:

Magalinskii, V. B. Angular momentum and parity conservation laws in the statistical theory of multiple production. Soviet Physics. JETP **36** (9) (1959), 67–69 (93–97 Ž. Èksper. Teoret. Fiz.).

"The general statistical method of the microcanonical distribution is employed to calculate the statistical weights of a many-particle system obeying an arbitrary statistics. The conservation laws for angular momentum and parity are taken into account. A general computational formula is obtained under the assumption that all particles obey Boltzmann statistics." (Author's summary)

D. J. Thouless (Birmingham)

5463:

Pavlikovskii, A.; and Shchuruvna, V. On the application of the method of supplementary variables in statistical physics. Soviet Physics. Dokl. **124** (4) (1959), 95–98 (69–71 Dokl. Akad. Nauk SSSR).

It is shown that if one uses the Bohm-Pines-Migdal-Galitskii method of redundant variables in the weakened form proposed by Kanazawa [Progr. Theoret. Phys. **18** (1957), 287–294; MR **19**, 711] to evaluate the partition function of a system of interacting particles one obtains an expression which in the limit $\hbar\to0$ leads to the classical expression for the partition function; this is not the case, if one uses the original Bohm-Pines-Migdal-Galitskii form.

D. ter Haar (Oxford)

5464:

Brueckner, K. A.; and Gammel, J. L. Properties of nuclear matter. Phys. Rev. (2) **109** (1958), 1023–1039.

This paper is devoted to the numerical determination of the properties of nuclear matter on the basis of the theory developed by the first author with a number of collaborators [the so-called Brueckner theory, see K. A. Brueckner, same Rev. **100** (1955), 36–45]. A new version of the theory is first presented, based on an infinite chain of equations relating single-particle energies and two-particle scattering matrices. Terminating the chain by introducing a suitable approximation the authors have obtained numerical solutions with the aid of the electronic computer IBM 704 (an appendix gives a procedure to replace the infinite chain by a single equation depending on an arbitrary parameter; this parametric equation was apparently found too late to be used in the numerical work). Various two-body potential of the Gammel-Thaler type were used. The quantities calculated are binding energy, equilibrium density, compressibility, effective mass at the Fermi surface and symmetry energy. The results given are in excellent agreement with the experimental values (if the neglect of the Coulomb energy is

taken into account). The uncertainties produced by the many theoretical and numerical approximations are only very partially discussed. *L. Van Hove* (Utrecht)

5465:

Klein, Abraham; and Prange, Richard. Perturbation theory for an infinite medium of fermions. Phys. Rev. (2) **112** (1958), 994–1007.

The theory of a Fermi gas of interacting particles is developed using the field-theoretical method of Green functions. These functions are defined as the expectation values of time-ordered products of wave operators for the state of physical interest (ground-state or temperature distribution). Except for the last section the paper treats the ground-state problem. The ground-state energy is expressed in terms of Green functions, the Feynman diagram analysis is carried out on the basis of the equations of motion for Green functions and the result is shown to be equivalent to previous treatments of the Fermi gas [in particular J. Goldstone, Proc. Roy. Soc. London Ser. A **293** (1957), 267–279; MR **18**, 975]. A heuristic discussion is given of the one-particle Green function, which leads to some properties of one-particle excitations.

L. Van Hove (Utrecht)

5466:

Prange, Richard; and Klein, Abraham. Generalized reaction matrix approach to the theory of the infinite medium of fermions. Phys. Rev. (2) **112** (1958), 1008–1020.

The Green function formalism of the Fermi gas in the ground-state, presented in the previous paper, is here used to formulate the essential approximations leading to the Brueckner theory of nuclear matter [see #5464 above] and the high density theory of the electron gas.

L. Van Hove (Utrecht)

5467:

Eyges, Leonard. Solution of Schrödinger equation for a particle bound to more than one spherical potential. Phys. Rev. (2) **111** (1958), 683–689.

The problem considered here is that of a particle moving in the field of n spherical potentials, each of which have a finite radius a_i and center $\mathbf{R}_i$. The method consists of expanding the solution inside each of these potentials in terms of the solutions of the single particle Schrödinger equation, and expanding outside these potentials in terms of a general solution of the free space Schrödinger equation. Finally joins of these inside and outside solutions are then made at each of the radii a_i. This leads to an infinite secular determinant for the energy E. The parameter determining the rate of convergence of the determinant is (a_i/a), where d is the distance to the nearest potential. The author gives an example of the application of the method in which the potential centers are at the corners of an equilateral triangle.

H. Feshbach (Cambridge, Mass.)

5468:

Bogolyubov, N. N.; and Solov'ev, B. G. On a variational principle in the many-body problem. Soviet Physics. Dokl. **124** (4) (1959), 143–146 (1011–1014 Dokl. Akad. Nauk SSSR).

Bogolyubov's extension of Fock's variational principle is applied to a system of interacting fermions. The coefficients in the canonical transformation of the Fermi amplitudes are determined from the condition that the energy expectation function be a minimum.

D. ter Haar (Oxford)

5469:

Khokhlov, Yu. K. On the moment of inertia of a many-particle system. I. Soviet Physics. JETP **36** (9) (1959), 203–206 (295–299 Ž. Eksper. Teoret. Fiz.).

The problem of the moment of inertia of a system of interacting particles is considered for the case where the Hamiltonian is invariant under rotation around at least one axis. An expression is obtained for the moment of inertia in terms of the (redundant) collective angle variable. It is shown that even in the case of a spherically symmetric system, the moment of inertia is different from zero. *D. ter Haar* (Oxford)

5470:

Bardeen, J.; Cooper, L. N.; and Schrieffer, J. R. Microscopic theory of superconductivity. Phys. Rev. (2) **106** (1957), 162–164.

This communication is in the form of a "Letter to the Editor" announcing a new theory of superconductivity. The theory is presented by the authors in more detail in a later publication [same Rev. **108** (1957), 1175–1204; MR **20** #2196]. The essential term in the Hamiltonian, leading to superconductivity, is an attractive interaction between electrons near the Fermi energy. It is necessary that this attractive interaction overcompensates the Coulomb repulsion between the electrons. The attraction between the electrons arises through the interaction of the electrons with the phonon field. A wave function is constructed in which electrons are paired, i.e., when an electronic state with number $\mathbf{k}$ is occupied by an electron with up spin, then the state with wave number $-\mathbf{k}$ is occupied by an electron with down spin. It is shown that this state is separated by an energy gap from all excited electronic states. The energy gap is of the type that may be expected to account for the electromagnetic properties of superconductors. The theory correctly predicts the experimentally observed isotope effect.

H. Statz (Waltham, Mass.)

5471:

Bogoljubov, N. N. On a new method in the theory of superconductivity. Nuovo Cimento (10) **7** (1958), 794–805. (Italian summary)

The author bases his work on a Hamiltonian given by Fröhlich which consists of a sum of one-electron energies and the interaction of these electrons with the lattice vibrations. Explicit Coulomb interaction is not introduced. The central feature of the paper is a canonical transformation of this Hamiltonian, eliminating certain divergences in the perturbation theory which is used for solving the eigenvalue problem. It is shown that there exists an "energy gap" for excitations from the ground state. Similarly, it is shown that a finite energy is also required for excitations from a state which carries a current, thus accounting for the existence of superconductivity. The paper arrives in a mathematically rather satisfactory way at essentially the same results as Bardeen, Cooper, and Schrieffer [see preceding review and MR **20** #2196].

H. Statz (Waltham, Mass.)

5472:

Nakamura, Ki-ichi. Two-body correlation of interacting fermions. Progr. Theoret. Phys. **21** (1959), 713–726.

The trial wave-function proposed by Bardeen, Cooper, and Schrieffer [Phys. Rev. (2) **108** (1957), 1175–1204; MR **20** #2196] for their theory of superconductivity is written in configuration space, and the expectation value of the energy is evaluated by methods similar to the cluster expansion method in the theory of an imperfect gas. The results are in agreement with those obtained by Bardeen, Cooper, and Schrieffer and by other workers.

D. J. Thouless (Birmingham)

5473:

Shirkov, D. V. On the compensation equation in superconductivity theory. Soviet Physics. JETP **36** (9) (1959), 421–424 (607–612 Ž. Eksper. Teoret. Fiz.).

A discussion is given of the connection between the energy levels of a system described in second quantization, the S-matrix and the energy operator. Using this connection the kernel of the integral equation used in the Bogolyubov-Tolmachev-Shirkov theory of superconductivity is expressed in terms of one-electron Green functions and the four-vertex Green functions of the pure Coulomb problem.

D. ter Haar (Oxford)

RELATIVITY

See also 5394, 5395, 5400, 5444.

5474:

★Aharoni, J. The special theory of relativity. Clarendon Press, Oxford, 1959. viii+285 pp. $7.20; 45s.

This textbook emphasizes those formal aspects of the Lorentz group most useful in contemporary theoretical physics, such as the representation theory of the Lorentz group, Lorentz-covariant variational principles, etc. The chapter headings are: (1) Space and time in inertial systems; (2) Three-dimensional tensors; (3) Maxwell's theory in tensor formulation; (4) General field theory; (5) Relativistic particle dynamics; (6) Elements of hydrodynamics; and (7) Spinors. Among the topics specially noted by the reviewer are the relationship between canonical and symmetric energy-stress tensor, and the treatment of both two-component and four-component spinor formalisms. The presentation stops short of modern quantum theory, though it is obvious that the author has slanted both the selection of subject matter and its presentation to prepare the student for a subsequent course in that field.

Being a textbook, the work avoids any literature references in the text. The twenty references collected at the end of the book are about half to books and review articles, half to research papers. The latter in no sense span the results discussed in the text. The book appears an excellent introduction to currently important Lorentz-covariant formalisms and, to a lesser extent, to the special theory of relativity itself (where it faces stiff competition from the many textbooks already on library shelves). It is not (and is not intended as) a research monograph or a review of the literature useful to the expert. It can be recommended both to students in theoretical physics and to mathematicians, even those who are unfamiliar with quantum mechanics and quantum field theory.

P. G. Bergmann (New York, N.Y.)

5475:

Frankl', F. I. Potential steady relativistic gas flows. Soviet Physics. Dokl. **123** (3) (1958), 1110–1112 (47–48 Dokl. Akad. Nauk SSSR).

In this very short paper the general case of a plane steady potential relativistic gas flow is discussed. The quantity

$$v^i = \frac{w}{\rho} u^i \quad (i = 0, 1, 2, 3),$$

where u^i is the 4-velocity, ρ the residual mass energy density and w the relativistic thermal function per unit proper volume is called the pseudo-velocity. In the case of potential flow v_i can be expressed as the gradient of a potential ϕ. For steady flow v_0 is constant and the ordinary velocity is determined from the equation

$$\hat{u}_i = -\frac{v_i}{v_0} \quad (i = 1, 2, 3).$$

This ordinary velocity is expressed in terms of a potential $\hat{\phi}$. The equation satisfied by $\hat{\phi}$ in Galilean coordinates is given and, in particular, for the case of a gas in which the velocity of sound is $(1/3^{1/2})c$ (c speed of light).

G. L. Clark (London)

5476:

Galli, Mario G. Vedute moderne circa i fondamenti delle trasformazioni di Lorentz. II. Riv. Mat. Univ. Parma **8** (1957), 313–335.

5477:

Kar, K. C. On linearisation of the relativistic Hamiltonian. Indian J. Theoret. Phys. **6** (1958), 65–67.

5478:

Bondi, H.; Pirani, F. A. E.; and Robinson, I. Gravitational waves in general relativity. III. Exact plane waves. Proc. Roy. Soc. London Ser. A **251** (1959), 519–533.

The authors define as a plane wave a solution of the vacuum field equations of general relativity that admits a 5-parametric group of motions (a "motion" being defined as a mapping of a Riemannian manifold on itself that leaves the metric unchanged; a group of motions is assured if the manifold possesses one or more fields of Killing vectors). They then proceed to show that a previously published solution [Bondi, Nature **179** (1957), 1072–1073] is the most general plane wave in existence, according to work by Petrov [Dokl. Akad. Nauk SSSR **105** (1955), 905–908; MR **18**, 101] and other papers by the same author. They examine both the mathematical and some of the physical properties of this plane-wave solution, utilizing in this discussion particularly "sandwich" waves, i.e., waves which are flat everywhere except between two parallel three-planes tangential to the light cone.

P. G. Bergmann (New York, N.Y.)

5479:

Pirani, F. A. E. Gravitational waves in general relativity. IV. The gravitational field of a fast-moving particle. Proc. Roy. Soc. London. Ser. A **252** (1959), 96–101.

It is shown that the field of a rapidly moving Schwarzschild particle resembles a plane wave [for definition of plane waves see preceeding review] at distances large compared to the Schwarzschild radius. This result is the analogue of the so-called Weizsäcker-Williams effect in electrodynamics [Williams, Mat.-Fys. Medd. Dansk. Vid. Selsk. **13** (1935), 1–50; v. Weizsäcker, Z. Physik **88** (1934), 612–625]. *P. G. Bergmann* (New York, N.Y.)

5480:

Takeno, Hyôitirô. A note on the theory of gravitational waves. Tensor (N.S.) **9** (1959), 73–75.

The author's failure [Tensor (N.S.) **6** (1956), 15–25; MR **18**, 704] to discover solutions of Einstein's vacuum field equations which (a) are non-flat, and (b) depend only on the coordinate difference $z-t$, is attributed to his assumption of (c) de Donder's coordinate conditions. [Solutions with properties (a) and (b) but not (c) have been known for many years; for literature see #5478 above.] *F. A. E. Pirani* (London)

5481:

Takeno, Hyôitirô. On geometric properties of some plane wave solutions in general relativity. I. Tensor (N.S.) **9** (1959), 76–93.

The first part of a tensor-geometrical study of solutions of the Einstein-Maxwell equations discovered by the author [Tensor (N.S.) **8** (1958), 59–70; MR **21** #2516b] and others [e.g. A. Lichnerowicz, C. R. Acad. Sci. Paris **246** (1958), 893–896; MR **19**, 1237 {to which the present reviewer did less than justice}]; these solutions represent combined gravitational and electromagnetic plane waves. Many interesting relations between the Riemann tensor, the Maxwell tensor, and vectors and tensors defined by them are exhibited. *F. A. E. Pirani* (London)

5482:

Rosen, Gerald. Geometrical significance of the Einstein-Maxwell equations. Phys. Rev. (2) **114** (1959), 1179–1181.

"Rainich geometries are analyzed in terms of the invariants associated with the Ricci vierbein of principal directions. At any point the four unit vectors of the vierbein pair off into two blades which contain the maxima and minima directions of mean curvature, respectively. The blades can 'mesh' into smooth integral surfaces for certain electromagnetic fields. In general, neighboring blades are shown to be related by only two independent differential conditions." (Author's summary)
D. W. Sciama (London)

5483:

Dirac, P. A. M. Fixation of coordinates in the Hamiltonian theory of gravitation. Phys. Rev. (2) **114** (1959), 924–930.

The author recapitulates his earlier work on the theory of the title [Proc. Roy. Soc. London Ser. A **246** (1958), 326–332, 333–343; MR **20** #724, #725], (modification: he now uses a metric with signature $+2$ instead of -2) and extends his general method to permit the introduction of additional weak equations into the theory. If the additional equations are $p_m \approx 0$, $m=1, \cdots, M$, then these momenta p_m and their conjugate coordinates may be eliminated entirely. In general, all the variables are retained, but the number of degrees of freedom is reduced, and Poisson brackets must be redefined [compare Dirac, Canad. J. Math. **2** (1950), 129–148; MR **13**, 306]. In the application to gravitation, the additional weak equations are just coordinate conditions, for which the author chooses (1) the maximal condition $g_{rs}p^{rs}=0$ to fix the surface $t=$constant ($r, s=1, 2, 3$; p^{rs} momenta conjugate to space metric g_{rs}), and (2) the conditions $\partial \bar{e}^{rs}/\partial x^s=0$ ($\bar{e}^{rs}=(\det g_{pq})^{1/3}e^{rs}$, $e^{rs}g_{st}=\delta_t^r$) to fix coordinates in the surface. The reduction cannot be carried out explicitly, but requires the use of approximation procedures, which are outlined. *F. A. E. Pirani* (London)

5484:

Anderson, James L. Factor sequences in quantized general relativity. Phys. Rev. (2) **114** (1959), 1182–1184.

"The problem of the order of factors in the constraint equations of general relativity has been investigated. Although the constraints plus Hamiltonian form a factor group in the nonquantized version of the theory, it does not follow directly that they will do so in the quantized version. The difficulty lies in the fact that the commutator of two expressions depends upon the order of factors chosen for them. A class of factor sequences is exhibited for the constraints such that they do indeed form a factor group." (Author's summary) *C. W. Kilmister* (London)

5485:

Fikhtengol'ts, I. G. On coordinate conditions in Einstein's gravitation theory. Soviet Physics. JETP **35** (8) (1959), 1018–1023 (1457–1465 Ž. Eksper. Teoret. Fiz.).

The author studies the conditions under which a set of equations, in Lagrangian form, are covariant with respect to a set of transformations. He then applies these conditions to the study of coordinate conditions for such equations. *M. Wyman* (Edmonton, Alta.)

5486:

Pratelli, Aldo M. Deduzione da un'unica azione delle equazioni indefinite e di contorno dei campi gravitazionale ed elettromagnetico. Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 203–221.

Largely expository discussion of variational principle for Einstein-Maxwell equations, including consideration of boundary conditions and surfaces of discontinuity. {The author's belief that derivation from a variational principle ensures compatibility of the equations is not necessarily justified.} The derivation of jump conditions from a variational principle seems first to have been given by Taub [Illinois J. Math. **1** (1957), 370–388; MR **19**, 816].
F. A. E. Pirani (London)

5487:

Skripkin, V. A. Conditions on discontinuity surfaces in the general theory of relativity. Soviet Physics. Dokl. **123** (3) (1958), 1144–1148 (799–802 Dokl. Akad. Nauk SSSR).

The author derives the Rankine-Hugoniot equations in general relativity and the conditions that the metric tensor and its derivatives must satisfy across a singular

hypersurface in space time. The results obtained are those given by the reviewer [Illinois J. Math. **1** (1957), 370–388; MR **19**, 816] and by O'Brien and Synge [Comm. Dublin Inst. for Adv. Study Ser. A. no. 9 (1953); MR **14**, 913]. The spherically symmetric case is discussed but the well-known fact that in the static case the coefficient of dr^2 in the line element, though continuous, may have a discontinuous derivative with respect to r is not mentioned. Such a discontinuity occurs when the Schwarzschild interior solution is fitted to his exterior one.

A. H. Taub (Urbana, Ill.)

5488:

Tulczyjew, W. Equations of motion of rotating bodies in general relativity theory. Acta Phys. Polon. **18** (1959), 37–55.

The method of Infeld [same Acta **13** (1954), 187–204; MR **16**, 531] is extended to the case of spherically symmetric rotating bodies. An energy-momentum density tensor constructed from the Dirac δ-function and its first derivatives

$$T^{\alpha\beta} = \sum_A (\overset{A}{t}{}^{\alpha\beta}(t)\delta(x-\overset{A}{\xi}(t)) - \overset{A}{t}{}^{\gamma\alpha\beta}(t)\delta_{,\gamma}(x-\overset{A}{\xi}(t)))$$

is assumed. The form of the $t^{\alpha\beta}$ and $t^{\gamma\alpha\beta}$ coefficients and the equations of motion (eqs. for $\xi^{\alpha}(t)$) of two bodies are derived from the integrability conditions of the field equations by the *E I H* approximation method. The corresponding Lagrange function is written down and the laws of conservation are investigated.

The equations of motion are integrated in some most important cases and a new relativistic effect is obtained, namely the motion of the line of equinoxes.

It is shown that the results obtained are dependent neither on the system of coordinates chosen for calculations nor on some arbitrariness in the used definition of the centres of mass of the bodies.

One must take into account some slight mistakes in this paper. Errata will appear in Acta Phys. Polon.

R. Michalska (Warsaw)

5489:

Coburn, N. A note on "The method of characteristics for a perfect compressible fluid in general relativity and non-steady Newtonian mechanics". J. Math. Mech. **8** (1959), 787–792.

Precise definitions of the terms "order of a quantity" and "order of a quantity as compared to another quantity," which were used in a previous paper [see same J. **7** (1958), 449–481; MR **20** #5623], are given. It is proved that the characteristic system in the non-steady Newtonian flow of a perfect fluid can be obtained from relativistic mechanics without making the assumption that the ratio of the local speed of sound to that of light is small.

G. C. McVittie (Urbana, Ill.)

5490:

Zel'manov, A. L. On the formulation of the problem of the infinity of space in the general theory of relativity. Soviet Physics. Dokl. **124** (4) (1959), 161–163 (1030–1033 Dokl. Akad. Nauk SSSR).

This paper recalls the fact that whether space is infinite or not depends in general on how space-time is decomposed into space and time. It also points out that a co-ordinate system may cover an infinite region without covering the whole of space-time. {It seems to the reviewer that the author's discussion would be clearer if the completeness of his various spaces were investigated.}

D. W. Sciama (London)

5491:

Marder, L. Flat space-times with gravitational fields. Proc. Roy. Soc. London. Ser. A **252** (1959), 45–50.

This paper describes the following exact solutions of Einstein's field equations. (i) A locally flat space-time outside a static infinite non-singular cylinder of everywhere positive density. (ii) A locally flat space-time outside a toroidal source.

The behaviour of the geodesics shows that both solutions represent gravitational fields although they are locally flat. The reason for this is that they do not have a Euclidean topology.

A modification of (i) shows that space-times which are not locally flat may be locally isometric and yet represent different gravitational fields because they are not homeomorphic. The author raises the important question whether this is possible for physically sensible sources, that is, ones which are non-singular, finite in extent, and everywhere of positive density.

D. W. Sciama (London)

5492:

Bouche, Liane. Les équations approchées du champ dans une théorie unitaire du type Einstein-Schrödinger. C. R. Acad. Sci. Paris **249** (1959), 1321–1323.

5493:

Knapecz, Géza. Notiz über die schwachen affinen Erhaltungssätze der Multimomente im Rahmen eines allgemeinrelativistisch-kovarianten Lagrange-Formalismus. Ann. Physik (7) **3** (1959), 340–344.

The term "multimoments" is intended to refer to expressions which result from sums of products of the coordinates and energy-like quantities (e.g., energy, angular momentum, moment of inertia are multimoments of order zero, one and two, respectively). Such multimoments and their conservation properties were studied recently by Mizkjewitsch [same Ann. **1** (1958), 319–333; MR **20** #722] within the framework of a theory based on an invariant Lagrangian density. In the present note a generalised common expression for all the multimoments of Mizkjewitsch is found in terms of generalised canonical momenta, and the corresponding affine equation of continuity is derived.

H. Rund (Durban)

5494:

★Catalano, Luciano R. La nueva física y el universo pentadimensional. [The new physics and the five-dimensional universe.] Published by the author, Buenos Aires, 1958. 215 pp.

ASTRONOMY

See also 5287, 5387.

5495:

Levy, Jacques. Sur les trajectoires des satellites proches. Bull. Géodésique (N.S.) no. 53 (1959), 7–20.

Secular perturbations and short-period perturbations in the orbital plane due to the second zonal harmonic of the earth's gravitational field are calculated by the method of the variation of the elements. The argument is the eccentric anomaly. Terms proportional to the square of the coefficient J of the second harmonic are not given.

The author considers that J is closely related to the dynamical ellipticity $(C-A)/C$, where C and A are respectively the moments of inertia about the polar axis and an equatorial axis. He considers that its relation to the geometrical ellipticity $(a-b)/a$, where a and b are respectively the major and minor axes of the meridian, is unknown. *J. A. O'Keefe* (Chevy Chase, Md.)

5496:

Brouwer, Dirk. (Editor) Proceedings of the Celestial Mechanics Conference, New York, N.Y. Astr. J. **63** (1958), 401–464.

Summary of 19 papers presented at the Conference, with a report on the round table discussion.

5497:

Contopoulos, G. Study of the potential in the plane of symmetry of a stellar system. Prakt. Akad. Athēnōn **31** (1956), 21–35. (Greek summary)

Dans son mémoire fondamental sur la dynamique des systèmes stellaires [Astrophys. J. **90** (1939), 1–154; **92** (1940), 441–642; MR **1**, 60; **3**, 216] S. Chandrasekhar a étudié le cas général d'un système à deux dimensions ayant un centre de symétrie et quelques cas spéciaux de systèmes à trois dimensions à symétrie axiale. Dans le présent mémoire, en supposant la validité de trois postulats donnés par S. Chandrasekhar, *Principles of stellar dynamics* [Chicago Univ. Press, Chicago, Ill., 1942; MR **4**, 57; p. 89; et cf. von der Pahlen, *Einführung in die Dynamik von Sternsystemen*, Verlag Birkhäuser, Basel, 1947; MR **10**, 333; p. 127], G. Contopoulos prouve les résultats pour deux dimensions par une nouvelle méthode, plus concise. Puis, par une extension de cette méthode, il étudie le potentiel sur le plan de symétrie d'un système stellaire à trois dimensions ayant à la fois un axe et un plan de symétrie. La force par unité de masse dans le plan de symétrie est donnée par la formule

$$\partial V/\partial w = -\varphi''\varphi^{-1}w+\varphi^{-4}wW(w^2/2\varphi^2),$$

où w est la distance à l'axe, où φ est une fonction du temps, et où $W(w^2/2\varphi^2)$ est susceptible suivant las cas de différentes formes simples qu'indique l'auteur. *M. Janet* (Paris)

5498:

Dedebant, Georges; et Schereschewsky, Philippe. Sur la possibilité d'observer la face cachée de la Lune. C. R. Acad. Sci. Paris **248** (1959), 3530–3532.

Consider the restricted problem of three bodies E, M and L, the Earth, the Moon and a Lunik (lunar rocket) respectively. The authors indicate an approximate solution corresponding to a shot at an angle $\varphi=LEM\approx 0$. A temporary observer, put at the libration point L_2, 55000 km beyond M, in the collinear configuration of the three bodies, is then in the position to observe the hidden side of the Moon. There exists beyond M a family of unstable elliptic orbits about L_2, leading to an orbit of ejection which hits the visible side of the Moon. *E. Leimanis* (Vancouver, B.C.)

5499:

★Mineur, H. La statistique stellaire. L'application du calcul des probabilités. Colloque tenu à Genève, 12–15 juillet 1939, pp.183–212. Collection Scientifique. Institut International de Coopération Intellectuelle, Paris, 1945. 276 pp. 10 francs suisses.

This paper gives a short review of the methods of stellar statistics. The first chapter deals with the distribution of stars in space. The effect of the absorption of light in space is considered and the chapter ends with the application to the problems of determining the structure of the galaxy and the distribution of the galaxies. The second chapter deals with the movements of the stars and contains a general account of Oort's theory of galactic rotation. The third chapter deals with stellar dynamics with a rather broad account of Lindblad's theory. *E. Lyttkens* (Uppsala)

5500:

McVittie, G. C.; and Wyatt, S. P. The background radiation in a Milne universe. Astrophys. J. **130** (1959), 1–11.

The integrated radiation at optical and radio frequencies from the distant unresolved sources in a Milne universe is calculated and compared with observation. The authors emphasise that similar calculations should be performed for other model universes, since although such calculations would involve lengthy numerical computation, the results may differ significantly from those for a Milne universe. *D. W. Sciama* (London)

5501:

Westfold, K. C. The polarization of synchrotron radiation. Astrophys. J. **130** (1959), 241–258.

The radiation from an electron moving at a speed which is a large fraction of that of light is calculated. The motion takes place in a uniform magnetic field of induction. The emitted radiation is elliptically polarized and the directions of the axes of the ellipse are computed. The spectral distribution of the emitted radiation is also worked out in detail. It is proved that the degree of polarization, in a given direction, of the radiation from a distribution of gyrating electrons increases steadily with frequency from the value $\frac{3}{4}$ asymptotically to unity. The intensity of the radiation emitted from the distribution of electrons will be affected if it traverses regions in which the magnetic field varies considerably in both magnitude and direction. The result will be a depolarization which can be estimated. *G. C. McVittie* (Urbana, Ill.)

GEOPHYSICS

See also 5376, 5495.

5502:

Bonnefille, R. Généralisation de la similitude de M. Vantroys en tenant compte de la variation des profondeurs marines. Houille Blanche **14** (1959), 547–555. (English summary)

5503:

Kohlsche, Kurt. Beitrag zur Frage der Entstehung von Vertikalbewegungen. Z. Meteorol. **12** (1958), 339–344.

Taking the individual time derivative $(d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla)$ of the vector equation of motion, the author derives an expression for d^2v_r/dt^2. Here $\mathbf{v}$ denotes the velocity, v_r its component along the radius vector r measured from the center of the Earth, and ∇ the gradient operator. The above quantity characterizes the evolution of the vertical accelerations and velocities which are responsible for the origin and growth of the vertical motions in the atmosphere. The author shows that the essential quantities which influence the above developments are the horizontal and total divergence of the velocity $\mathbf{v}$ and its vertical gradient, the vertical change of the horizontal wind, the non-geostrophic part of the North-South component of the wind, and the heat sources.

E. Leimanis (Vancouver, B.C.)

5504:

Rinner, Karl. Koeffizientenbedingungen in Potenzreihen für konforme Abbildungen des Erdellipsoides in die Ebene. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1958**, 51–72.

Consider the conformal mapping defined by $X+iY = F(q+il)$ where (q, l) are isometric (Mercator) coordinates on an ellipsoid of revolution. The coefficients of the power series of $X=F_1(q, l)$, $Y=F_2(q, l)$ are mutually related through the conditions imposed by the Cauchy-Reimann and the Laplace equations. The author explicitly lists these relations up to the fifth order, and extends the results to include the expansion of q in terms of the geographic latitude, and the corresponding inverse series. These results are special cases (but carried out to a larger number of terms) of the reviewer's paper [Boll. Geodes. Sci. Affini **11** (1952), 379–394; MR **18**, 978].

B. Chovitz (Washington, D.C.)

OPERATIONS RESEARCH, ECONOMETRICS, GAMES

See also 5265, 5523.

5505:

★**Divisia, F. Role de la statistique dans les nouveaux problèmes de l'économie politique.** L'application du calcul des probabilités. Colloque tenu à Genève, 12–15 juillet 1939, pp. 213–257. Collection Scientifique. Institut International de Coopération Intellectuelle, Paris, 1945. 276 pp. 10 francs suisses.

The article contains general comments and some critical remarks on the use of statistics and probability calculus in the field of economic research.

T. Haavelmo (Oslo)

5506:

★**Polak, J. J. Études statistiques de la structure économique.** L'application du calcul des probabilités. Colloque tenu à Genève, 12–15 juillet 1939, pp. 259–275. Collection Scientifique. Institut International de Coopération Intellectuelle, Paris, 1945. 276 pp. 10 francs suisses.

An expository article intended to illustrate the principles of econometrics.

T. Haavelmo (Oslo)

5507:

McKenzie, Lionel W. On the existence of general equilibrium for a competitive market. Econometrica **27** (1959), 54–71.

The author generalizes earlier work of Arrow and Debreu [Econometrica **22** (1954), 265–290; MR **17**, 985] and himself [ibid. 147–161] on sufficiency conditions for the existence of competitive equilibrium. To quote the author, the directions of generalization are: "First, there are no restrictions on the dimensionality of the production and consumer sets. ... Second, the assumption that there are always desired goods and always productive goods is replaced by the assumption that the economy ... cannot be divided into two groups of consumers where one group is unable to supply any goods which the other group wants. ... A third relaxation is that production processes may be reversible. ... The argument uses the most elementary of the fixed point theorems, that of Brouwer, without introducing additional complications". The author has succeeded admirably in eliminating some of the more artificial assumptions of earlier analyses.

K. J. Arrow (Stanford, Calif.)

5508:

Roy, A. D. The valuation of random income streams. Metroecon. **10** (1958), 136–154.

For each possible realization of a stream of random income payments, a capital value, equal to the sum of the payments discounted at a suitable rate of interest, can be found. The author investigates in particular the case where the random income payments are generated by a Markov process and finds the mean and variance of the capital value. To study the problem of choice among random income streams, he then considers a vector of random income payments generated by a Markov process and presents formulas for the mean vector and the covariance matrix for the valuations (the latter has an obvious error of sign). He then considers applications to specific investment problems.

K. J. Arrow (Stanford, Calif.)

5509:

Solow, Robert M. Competitive valuation in a dynamic input-output system. Econometrica **27** (1959), 30–53.

The author examines the price implications of a dynamic input-output model. Let A be the matrix of current input coefficients, B that of capital coefficients, and D the vector of final demands. Then the output vector x_t must satisfy the relations, $Bx_{t+1} = (B+C)x_t - D$, where $C = I - A$. Under competitive assumptions the prices, p_t, and rate of interest, r_t, must satisfy the relation, $(B+C)'p_{t+1} = (1+r_t)B'p_t + W$, where W is the vector of direct labor costs. By examining the nature of the solutions, it is made plausible that the existence of a stable solution to one

equation is associated with an unstable solution to the other.

He then remarks that the requirement of equality everywhere in the above requires special assumptions about full employment of all factors, in particular of accumulated capital stocks from the past. Instead the problem can be restated as that of maximizing the total value of Nth period stocks for some fixed N at arbitrary valuations. This is a linear programming problem, and the interpretation of the dual solution is presented in some detail. *K. J. Arrow* (Stanford, Calif.)

5510:

Dobbie, James M. On the allocation of effort among deterrent systems. Operations Res. 7 (1959), 335–346.

"The capability to retaliate should war be initiated by our enemy is used as the deterrence criterion. Assuming that the enemy's blunting effort can be allocated among our deterrent systems with full knowledge of our distribution, what is the best allocation of our effort? This problem is solved in detail for the case in which the alternatives consist of a system of fixed bases and a system of mobile units." [From the author's summary]

J. Kiefer (Ithaca, N.Y.)

5511:

★Friedrich, Peter. Die Variationsrechnung als Planungsverfahren der Stadt- und Landesplanung. Abhandlungen der Akademie für Raumforschung und Landesplanung, Bd. 32. Walter Dorn Verlag, Bremen-Horn, 1956. 58 pp. Paperbound: DM 12.00.

The calculus of variations is applied to determine the optimum (with respect to cost) configuration of road traffic nets. A number of affecting factors, viz., the density of traffic sources, the shape of the settled areas, the shape of the road net, and the dimensions of the mesh of the net, are considered. These are expressed as numerically measurable functions which combine to form a cost integral to be minimized.

B. Chovitz (Washington, D.C.)

5512:

Levi, Eugenio. L'interpolazione di tavole di sopravvivenza mediante somme di funzioni esponenziali. Statistica. Bologna **19** (1959), 3–19.

Several methods are described and illustrated for fitting to the survivorship function l_x in a mortality table an expression of the form $\sum_{r=0}^{k-1} b_r u_r^x$. It is shown how the fitting of such an expression facilitates the calculation of present values of life annuities.

T. N. E. Greville (Kensington, Md.)

5513:

Saaty, Thomas L. Coefficient perturbation of a constrained extremum. Operations Res. 7 (1959), 294–302.

If the data of a linear programming problem are functions of a parameter, t, the solution vector will also be a function of t and the problem can be written in the form: Find the function of t, $\mathbf{x}(t)$, such that for each t the function $v(t)=\mathbf{c}(t)\mathbf{x}(t)$ is minimized subject to the constraints $\mathbf{A}(t)\mathbf{x}(t)\geqq\mathbf{b}(t)$, $\mathbf{x}(t)\geqq\mathbf{0}$. Noting that the simplex criteria will also be functions of t the problem is solved in the following steps: (1) solve in the usual way for an arbitrary value of t; (2) from the simplex criteria, expressed as functions of t, determine the range of t within which the basis just found is feasible and optimal; (3) choose a value of t just outside this range and solve the problem for this new value; repeat steps (2) and (3) until optimal feasible bases have been found for all desired values of the parameter. Since only a finite number of optimal bases exist, the solutions for the whole range of t can be found in a finite number of repetitions of these steps. This procedure is an extension of the parametric programming methods developed principally by Saaty and Gass [J. Operations Res. Soc. Amer. **2** (1954), 316–319; MR **16**, 51] and A. S. Manne [*Scheduling of petroleum refinery operations*, Harvard University Press, Cambridge, 1956].

The sensitivity of the extreme value of a linear programming problem to changes in the data, or errors in them, is ascertained by regarding the data as functions of some parameter and calculating the total derivative of v with respect to the parameter.

R. Dorfman (Cambridge, Mass.)

5514:

Wolfe, Philip. The simplex method for quadratic programming. Econometrica **27** (1959), 382–398.

The standard quadratic programming problem is the following: find a non-negative vector $x\ (n\times 1)$ which minimizes the function $f(\lambda, x)=\lambda px+\frac{1}{2}x'Cx$ subject to the constraint $Ax=b$, where $A\ (m\times n)$, $b\ (m\times 1)$, C ($n\times n$, symmetric), $p\ (1\times n)$ are preassigned matrices and vectors, and $\lambda\geqq 0$ is a preassigned scalar. If C is positive definite, the problem can be solved by a slight modification of the simplex method for linear programming. It follows from the Kuhn-Tucker theorem for nonlinear programming that a sufficient condition for x to be a solution is that there exist vectors v ($n\times 1$, non-negative) and u ($m\times 1$, unrestricted) such that x, u, v satisfy (1) $Ax=b$, (2) $Cx-v+A'u=-p'\lambda$, (3) $x\geqq 0$, (4) $v\geqq 0$, (5) $v'x=0$. Restrictions (1)–(4) constitute a linear inequality problem and, as is well known, can be solved by the simplex method (if a solution exists) by constructing and solving an appropriate linear programming problem. Because of restriction (5) the simplex routine must be modified as follows: Choose an initial basic solution in which $v=0$. Then, in the iterative procedure never permit a component of v to become positive if the corresponding component of x is positive and vice versa. This algorithm will lead to vectors x, u, v that satisfy (1)–(5) so that x is a solution to the quadratic programming problem.

This procedure can be modified to be of use when C is positive semidefinite and to the parametric case where solutions are desired for the full range of λ, $0\leqq\lambda\leqq\infty$. Because this is a minor modification of the simplex procedure it is adapted to machine computing, and quadratic programming problems with as many as ninety constraints have been solved. An example is given.

R. Dorfman (Cambridge, Mass.)

5515:

★Deutsche Statistische Gesellschaft. Anwendungen der Matrizenrechnung auf wirtschaftliche und statistische Probleme. Einzelschriften der Deutschen Statistischen Gesellschaft, Nr. 9. Physica-Verlag, Würzburg, 1959. 262 pp.

A collection of expository articles commissioned by the German Statistical Society as an introduction to and illustration of uses of the matrix calculus. These consist

of (1) a 64 page standard introduction to matrices, systems of linear equations and the eigenvalue problem (A. Klingst); (2) a concise description, with examples, of the following numerical methods: the Gaussian algorithm for matrix inversion or solution of systems of linear equations; the iterative Gauss-Seidel method; the method of K. Hessenberg [Doctoral dissertation, T. H. Darmstadt, 1941] for determining all eigenvalues and eigenvectors of a matrix the principal step of which is transforming the matrix to the Cayley-Hamilton standard form; and the evaluation of a single characteristic root of largest absolute value and its eigenvector by iteration (H. Scholz); (3) a sketch of input-output analysis (A. Klamecker) with special reference to its use within a firm (K. Wencke) and a detailed account of the use of open input-output models in standard costing and production planning by calculation of throughputs of basic materials required to sustain the production of a desired bill of products of an integrated large firm (O. Pichler); (4) an exposition of the basic theory of Linear Programming, Game Theory, and Input-Output Analysis (W. Wetzel); and of the simplex method of computation (F. Ferschl); (5) applications to multivariate statistical analysis including the theorem of Cochran on the distribution of quadratic forms of given rank [Proc. Cambridge Philos. Soc. **30** (1934), 178–191] (J. Roppert) and tri-variate contingency tables (A. Adam).

M. J. Beckmann (Providence, R.I.)

5516:

Mills, Edwin S. A note on the asymptotic behavior of an optimal procurement policy. Management Sci. **5** (1959), 204–209.

Let x_i be the amount required in the ith period, z_i the amount procured, $c(z)$ the cost of procurement (assumed quadratic with increasing positive marginal costs), and r the unit cost of holding inventory. Modigliani and Hohn [Econometrica **23** (1955), 46–66; MR **16**, 733] have studied for fixed N the problem of minimizing, $\sum_1^N [c(z_j)+rI_j]$, where I_j is the inventory resulting from the policy and is restricted to be non-negative. The author supposes that requirements are growing geometrically at a rate $\rho-1$. Suppose that the firm in each period makes a decision which is the optimal first period procurement of a new N-period program. Then it is shown that inventory converges to a level independent of its initial value and at this limit the decision rule takes the simple form of procuring an amount proportional to current requirements, the factor of proportionality being, $\rho^N/(N\rho-N+1)$, independent of costs. *K. J. Arrow* (Stanford, Calif.)

5517:

★**Vickrey, William. Self-policing properties of certain imputation sets.** Contributions to the theory of games, Vol. IV, pp. 213–246. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

This paper is devoted to the introduction of some stability conditions for von Neumann-Morgenstern solutions to games. Let a given set of imputations be regarded, for social reasons, as "conforming". A "heresy" is an imputation which dominates a conforming imputation c and is itself non-conforming. A "policing imputation" is a conforming imputation which dominates a heresy. A member of the coalition which enforced the heresy is said to be "penalized" if he is worse off under the policing imputation. If all policing imputations which police a given heresy penalize a given member j of the heretical coalition, then the heresy is "suicidal" for j. If every heresy from a given conforming imputation c is suicidal for some heretic, then c is a "strong imputation" (relative to the set of conforming imputations). If every conforming imputation is strong, the set is a "self-policing pattern". Finally, a solution (in the von Neumann-Morgenstern sense) which is also a self-policing pattern is a "strong solution".

The bulk of the results are related to the existence of strong solutions. With regard to simple games, it is shown that homogeneous majority games possess a unique strong solution, while non-homogeneous majority games possess no strong solution. For these definitions see J. von Neumann and O. Morgenstern, *Theory of games and economic behavior* [Princeton Univ. Press, 1944; MR **6**, 235; pp. 433–435]. The possibility of strong solutions in four-person constant-sum games is then studied; it is shown that there exist none for games in a certain neighborhood of the center. *K. J. Arrow* (Stanford, Calif.)

BIOLOGY AND SOCIOLOGY

5518:

★**Haldane, J. B. S. Mathematical study of heredity: Application of the theory of probability to the study of heredity.** L'application du calcul des probabilités. Colloque tenu à Genève, 12–15 juillet 1939, pp. 129–148. Collection Scientifique. Institut International de Coopération Intellectuelle, Paris, 1945. 276 pp. 10 francs suisses.

A brief summary of genetical statistics as it was in 1939, dealing with such topics as significance tests, χ^2, fitting of truncated binomials, u-scores for linkage testing, estimation of linkage by inverse probability and maximum likelihood, and the effect of random fluctuations in gene frequencies. *C. A. B. Smith* (London)

5519:

Taga, Yasushi; and Isii, Keiiti. On a stochastic model concerning the pattern of communication: Diffusion of news in a social group. Ann. Inst. Statist. Math. Tokyo **11** (1959), 25–43.

A social group homogeneous in time and in person is defined as one in which a certain piece of information will be transmitted with equal probability from any "knower" to any other person and in any time interval $(t, t+\Delta t)$. Besides this process of inner transmission, an outside source is assumed, which bombards the population with the same piece of information, so that every individual can receive it in any time interval with equal probability. The probability densities ("intensities") of inner transmission and receipt from the outside source are denoted by η and μ, respectively.

The stochastic process results in a tree, whose nodes are the "knowers" and by which the genealogy of the knower's knowledge can be traced to the outside source

from which the dissemination started. The state of the tree is represented by the random variable $U(t)$, all isomorphic directed graphs being represented by the same state of that variable.

The authors work primarily with three random variables, namely, $N(t)$, the number of knowers; $K(t)$, the number of knowers who had not yet passed the information on; and $L(t)$, the number of knowers who received the news directly from the source. They derive closed expressions or recursive formulas for several probability distributions, expected values, and variances.

Of particular interest are the following results. (1) The conditional probabilities $p_{n,k} \equiv \Pr\{K(t)=k|N(t)=n\}$ and $p_{n,l}^* \equiv \Pr\{L(t)=l|N(t)=n\}$ are independent of time t and the size of the group. (2) For sufficiently large n, $p_{n,k}$ and $p_{n,l}^*$ are asymptotically normal distributions. (3) The bi-variate statistic $\{N(t), L(t)\}$ is a sufficient statistic for the family of distribution $\Pr(U(t); \lambda, \mu)$. A procedure for estimating λ/μ is given.

The following applications are proposed. (1) If the hypothesis H_0 that the group is homogeneous be true, then, if the outside source stops its function as soon as the first person has received the information, the conditional expectation $E\{K(t)|N(t)=n\}$ and the corresponding conditional variance should be respectively $E_n(K)=n/2$; $V_n(K)=n/12$. (2) Assuming homogeneity, if several groups with λ_i all equal $(=\lambda)$ can be found, the estimates of μ_i/λ allow quantitative estimates of the effectiveness (intensities) of several outside sources. (3) The stochastic behavior of $U(t)$ can be evaluated constructively, based on the knowledge of μ/λ and the homogeneity of the social group. *A. Rapoport* (Ann Arbor, Mich.)

5520:

★**Spearman, C. Role of statistics in investigation of laws of psychology.** L'application du calcul des probabilités. Colloque tenu à Genève, 12–15 juillet 1939, pp. 149–160. Collection Scientifique. Institut International de Coopération Intellectuelle, Paris, 1945. 276 pp. 10 francs suisses.

The author reviews certain mathematical and statistical "laws" in psychology, such as Weber's and Fechner's laws on the threshold and magnitude of sensation, the law of learning, the law of constant output, hierarchical systems and factor analysis. The treatment is very concise, pungent, and elementary. *C. A. B. Smith* (London)

5521:

Wolins, Leroy. An improved procedure for the Wherry-Winer method for factoring large numbers of items. Psychometrika **24** (1959), 261–264.

A technique is given differing from a previous one presented by the same author [Psychometrika **18** (1953), 161–179] in that the use of variance terms is eliminated from the computations.

INFORMATION AND COMMUNICATION THEORY

See also 4904, 5264, 5265, 5291, 5527a-b.

5522:

★**Meyer-Eppler, W. Grundlagen und Anwendungen der Informationstheorie.** Kommunikation und Kybernetik in Einzeldarstellungen, Bd. 1. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. xviii+446 pp. (1 plate). DM 98.00.

"Ich habe versucht, in dem vorliegenden Buch im wesentlichen der umgangssprachlichen Bedeutung des Wortes "Information" Rechnung zu tragen, wobei jedoch durch eine exakte Definition aller mit "Information" zusammengesetzten Wörter (Informationsgehalt, Informationsdichte, Informationsvolumen usw.) dafür gesorgt werden musste, dass die mathematische Behandlung eine feste Basis erhielt.

Zentrales Anliegen aller Betrachtungen ist die menschliche Kommunikationskette (Kap. 1) und der in ihr stattfindende Zeichenverkehr, der von Signalen getragen wird, die den Sinnesorganen zugänglich sind. Die messbaren Eigenschaften dieser Signale bilden die Grundlage für alle weiteren Untersuchungen (Kap. 2), wie etwa für die Frage nach den zur Signalübermittlung geeigneten Übertragungssystemen (Kap. 3), die Statistik der hierbei verwendeten stereotypen Signalformen ("Symbole") (Kap. 4) und den Einfluss von Störungen auf die Signalübermittlung (Kap. 5) sowie die mögliche Sicherung gegen Übertragungsfehler (Kap. 6). In Kap. 7 tritt der informationsempfangende Kommunikationspartner mit seinen Sinnesorganen in Erscheinung, zunächst als Empfänger von Signalen und von Kap. 8 ab als Empfänger von Zeichen. Als die wichtigsten Zeichenträger werden in Kap. 9 die akustischen und optischen Valenzklassen behandelt. Von hier aus ergibt sich ein unmittelbarer Zugang zur höchsten Stufe menschlicher Kommunikation, zur sprachlichen Kommunikation. Im Anschluss an die Probleme und Methoden der strukturellen Linguistik (Kap. 10) ist das letzte Kapitel der realen Sprachübermittlung gewidmet, d.h. dem Schicksal der Sprachzeichen in einem zwischen dem sende- und dem empfangsseitigen Kommunikationspartner etablierten gestörten Übertragungskanal. Durch eine genügende Zahl von Hinweisen wurde dafür gesorgt, dass jedes Kapitel zur Not auch ohne die vorangegangenen Kapitel verständlich ist." (From author's preface) *J. Wolfowitz* (Ithaca, N.Y.)

5523:

★**Adam, A. Messen und Regeln in der Betriebswirtschaft: Einführung in die informationswissenschaftlichen Grundzüge der industriellen Unternehmensforschung.** Physica-Verlag, Würzburg, 1959. viii+179 pp. DM 27.00.

An elementary introduction to information theory with emphasis on its applicability to operations research. *M. J. Beckmann* (Providence, R.I.)

5524:

Levine, Jack. Variable matrix substitution in algebraic cryptography. Amer. Math. Monthly **65** (1958), 170–179.

The concept of disguising text by a linear transformation was introduced in two papers by Lester S. Hill [same Monthly **36** (1929), 306–312; **38** (1931), 135–154]. Hill's transformation can be represented by $c=Ap$, where A is a nonsingular $n\times n$ matrix over a ring containing the elementary marks from which the text is composed, and p and c are vectors. The vector p is a portion of the text to be concealed, for example it could be n letters. The

vector c is to be transmitted. If much text were disguised by this method the transformation A would be exposed to extensive study.

The subject of the present paper is transformations which change continually, concealing the text by an operation which is itself only momentarily exposed. The author proposes two versions. One has the form $c_i = Ap_i + Bk_i$ where A and B are $n\times n$ matrices and k_i is a vector which is changed with each application of the transformation. A second version has the matrix $A(t)$ as a function of a parameter t, and then uses $c_i = A(t_i)p_i$. Of course $A(t)$ must have an inverse for all admissible t.

Conditions are given that each of the two versions should be involutory, and methods of finding matrices with the needed properties are given. A passing allusion is made to "computational difficulties". It seems to the reviewer that this is a critical point, and these difficulties will prevent these schemes from being of any practical importance until such time as computers begin to send secret messages to one another.

H. H. Campaigne (Jessup, Md.)

5525:

Sakaguchi, Minoru. Notes on statistical applications of information theory. IV. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **6**, 54–57 (1959).

[For parts I–III see same Rep. **1** (1952), 27–31; **4** (1955), 57–68; **5** (1957), 9–16; MR **14**, 996; **17**, 758; **19**, 896.] The author has defined the capacity of an experiment as the maximum of the rate for all possible prior densities,

$$C(f_1, \cdots, f_n) = \max_{\zeta} \sum_{i=1}^{n} \zeta_i \int f_i(x) \log \frac{f_i(x)}{\sum_{i=1}^{n} \zeta_i f_i(x)}\, dx,$$

where $f_i(x)$ $(i=1, 2, \cdots, n)$ are generalized densities of a homogeneous family of probability measures. If there exists a convex linear combination $\sum_{i=1}^{n} \zeta_i^* f_i(x)$ $(i=1, 2, \cdots, n)$ with positive coefficients ζ_i^* such that

$$\int f_k(x) \log \frac{f_k(x)}{\sum_{i=1}^{n} \zeta_i^* f_i(x)}\, dx$$

is independent of k $(k=1, \cdots, n)$, then the probability n-vector $\zeta^* = (\zeta_1^*, \zeta_2^*, \cdots, \zeta_n^*)$ yields $C(f_1, \cdots, f_n)$. Let $p_{i\cdot} = (p_{i1}, p_{i2}, \cdots, p_{in})$ $(i=1, \cdots, n)$ be n probability n-vectors which are linearly independent. Suppose there exists a probability n-vector ζ^* such that

$$\sum_{i=1}^{n} \zeta_i^* p_{ij} = \exp(-X_j)(\sum_j \exp(-X_j))^{-1} \quad (j = 1, 2, \cdots, n)$$

and $\zeta_i^* > 0$ $(i=1, \cdots, n)$, where the vector $X = (X_1, \cdots, X_n)$ is defined by $\sum_{j=1}^{n} p_{ij} X_j = H_i$ $(i=1, 2, \cdots, n)$ and $H_i = -\sum_j p_{ij} \log p_{ij}$; then ζ^* yields the capacity $\log(\sum_1^n \exp(-X_j))$. Consider density functions of an exponential family $f(x|\theta) = \beta(\theta)e^{\theta x} r(x)$, where $r(x) \geq 0$ and $\beta(\theta)$ is defined by $\int f(x|\theta)dx = 1$. If a unique positive number ζ^* can be found such that

$$\int f(x|\theta_1) \log \frac{f(x|\theta_1)}{f(x|\bar\theta)}\, dx = \int f(x|\theta_2) \log \frac{f(x|\theta_2)}{f(x|\bar\theta)},$$

where $\bar\theta = \zeta^*\theta_1 + (1-\zeta^*)\theta_2$, then the common amount of information is defined as the pseudo-capacity of the dichotomous experiment composed of $f(x|\theta_1)$ and $f(x|\theta_2)$. Results are given for the pseudo-capacity of an exponential family, and a limiting value as the parameters θ_1 and θ_2 tend to θ. The results are applied to compute the appropriate values for the binomial, Poisson, normal, and exponential distributions. The author does not mention related results for binary channels given by Muroga [J. Phys. Soc. Japan **8** (1953), 484–494; MR **15**, 450], Silverman [Trans. I.R.E. **IT-1** (1955), 19–27], and Sze-hou Chang [ibid. **IT-4** (1958), 152–159]. The equations for ζ_1^* and ζ_2^* at the bottom of page 55 are set up incorrectly.

S. Kullback (Washington, D.C.)

SERVOMECHANISMS AND CONTROL

See also 4917, 4918, 5047, 5053.

5526:

Cutteridge, O. P. D. The stability criteria for linear systems. Proc. Inst. Elect. Engrs. C **106** (1959), 125–132.

"It is shown that the various stability criteria for linear systems can be conveniently obtained, and interrelated, by means of continued fractions. A new canonical form for a Hurwitz polynomial as a product of continued fractions is also given, together with an alternative set of determinantal criteria with determinants of about half the order of the Hurwitz determinants."

Author's summary

5527a:

de Troye, N. C. Classification and minimization of switching functions. Philips Res. Rep. **14** (1959), 151–193. (French and German summaries)

5527b:

de Troye, N. C. Classification and minimization of switching functions. Philips Res. Rep. **14** (1959), 250–292.

Two boolean functions of the same number of boolean variables are said to be equivalent if they can be derived from each other by permutations and negations of variables. The minimum diode problem is to represent a boolean function as a minimum (with as few literals as possible) sum of products. A complete table of equivalence classes is given for boolean functions of up to four boolean variables. A table of solutions of the minimum diode problem is given for up to four variables. Tables useful for the minimum diode problem with up to six variables are given. This class of problems has been recently discussed from an algebraic-topological point of view by J. P. Roth [Trans. Amer. Math. Soc. **88** (1958), 301–326; MR **20** #3755; and the bibliography there cited].

S. Sherman (Philadelphia, Pa.)

5528:

Ioanin, Gh. Sur un type de problèmes concernant les schemas à selecteurs. An. Univ. "C. I. Parhon" Bucureşti. Ser. Acta Logica **1** (1958), no. 1, 187–193. (Russian and English summaries)

The author presents a method of synthesizing a sequential switching circuit consisting of relay contacts and a selector (i.e., rotary switch). The method is illustrated by means of an example in which the rotor of the selector is to be advanced one position each time a button switch is

pressed. When the rotor has completed a revolution, the circuit is to be in its initial condition. The required sequential operation of the selector is specified by a recursion equation. The method consists of determining the recursive functions which describe the sequence of states of the intermediate (secondary) relays and the relay actuating the main selector contact. These functions are Boolean expressions describing the contact network. The number of secondary relays (states) is also determined in the process. *E. K. Blum* (Los Angeles, Calif.)

BIBLIOGRAPHICAL NOTES

Proceedings of Vibration Problems. This publication of the Institute of Basic Technical Problems in the Polish Academy of Sciences is devoted to problems of dynamics and vibrations in acoustics, electricity and mechanics and to problems of coupled electro-mechanics, thermal and mechanical fields. The first issues will appear irregularly but in 1961 the publication will become a quarterly, to which it will be possible to subscribe. The price of the first issue, dated Warsaw 1959, is zł. 18. Copies may be ordered from Ars Polona, Przedmieście 7, Warsaw, or obtained on scientific exchange, from Department of Vibrations IBTP, Świętokrzyska 21, Warsaw.

Technology Reports of the Kansai University. This is a publication in English of the Faculty of Engineering, Kansai University, Osaka, Japan. Vol. 1, no. 1 is dated March 1959. The first issue contains sections on Mechanical, Chemical and Metallurgical Engineering, with an article of mathematical interest in the first section.

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